EXERCISES
GEOMETRY REU
SUMMER 2012-2013

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Problems:

(1) Draw a triangulation of a genus two surface with one boundary component. Compute the Euler characteristic.

(2) For each of the following surfaces, draw the surface in the usual form, and then draw a sequence of pictures showing how to expand the boundary and make the surface into a fatgraph (skinny surface). There is more than one right answer, since for each surface there are many ways to draw it as a fatgraph.
  (a) A once-punctured torus (we did this in class)
  (b) A thrice-punctured sphere
  (c) A genus two surface with one boundary component
  (d) A genus one surface with two boundary components
  (e) A genus one surface with arbitrarily many boundary components

(3) Build some fatgraphs by just gluing some rectangles together. Compute the Euler characteristic and genus of the surfaces you just built.

(4) Carefully write up the proof I outlined in class that for a fatgraph S, we have $\chi(S) = J - R$, i.e. the number of junctions minus the number of rectangles. Use pictures.

(5) Number the junctions in a fatgraph S by $j = 1, 2, \ldots, J$. Let k_j be the valence (number of attached edges) of junction j in S. Show that

$$\chi(S) = \sum_{j=1}^{J} \frac{2 - k_j}{2}$$

(6) Show that any triangulation of a triangle has $V - E + F = 1$. Hint: add the triangles one at a time.

(7) Expand out and check that $[x, y]^3 = [xyx^{-1}, y^{-1}xyx^{-2}][y^{-1}xy, yy]$.

(8) Compute $\mathrm{cl}(abABAbaB)$.

(9) Show that $\mathrm{cl}(yxy^{-1}) = \mathrm{cl}(x)$ (and thus $\mathrm{scl}(yxy^{-1}) = \mathrm{scl}(x)$) for all $x \in [F, F]$ and $y \in F$.

(10) Find all the possible pieces (triangles and rectangles) that can appear in a fatgraph with boundaries (powers of) $abAB$.

(11) For any $w \in F$, if y is a w-word and $x \in F$, prove that xyx^{-1} is a w-word. You'll want to use the fact that a w-word is the image of w until some homomorphism $F \to F$.

(12) Compute the defect of $H_{BA} + H_{aB} + H_{Ab} + H_{ba}$.

(13) Is $\overline{H}_{ab}$ extremal for $[a, b]$? (Remember $D(\overline{H_\sigma}) = 2D(H_\sigma)$; this is specific for counting quasimorphisms).

Problems that we don't know the answer to:

(1) Find other formulas like $[x, y]^3 = [xyx^{-1}, y^{-1}xyx^{-2}][y^{-1}xy, yy]$.

(2) Find a general formula for $\mathrm{cl}([a, b]^n)$. (Solved! It's $\lfloor \frac{n}{2} \rfloor + 1$).

(3) Write a program to compute $\mathrm{cl}(x)$ using the algorithm that I describe in the slides, probably using trimmed branching or something. It seems possible to me this would be faster than integer programming. You should probably talk to me before you start this project.

(4) Find formulas for $\ell(w^n|w)$ for $w \in F$, or rather, find algebraic formulas relating powers or w to products of w-words. One way to do this would be to exhaustively enumerate small products of w-words and see which ones are powers of other w-words.

(5) Recall the argument for the formula for $\mathrm{cl}([a,b]^n)$. Try to use some argument like this to compute $\mathrm{cl}(a^n b^n (AB)^n)$.