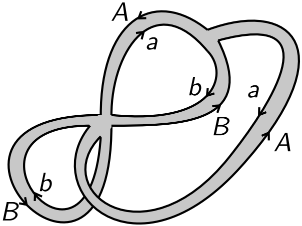
# Surfaces and commutators
# (Geometry REU)
# Class 4

Alden Walker
(Later: Danny Calegari)

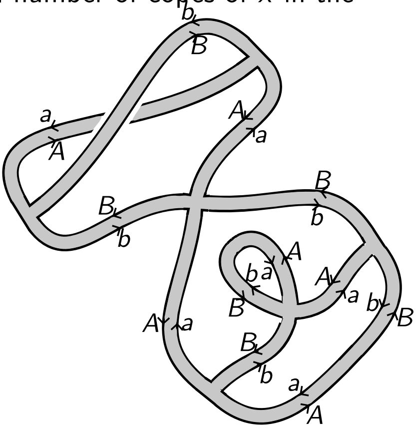
Summer 2013

# Review (Efficient surfaces)

Instead of searching for the smallest-genus surface with one boundary $x$, we search for the most *efficient* surface, with possibly many boundary components. A surface is efficient if it minimizes $\frac{-\chi(S)}{2n(S)}$, where $n(S)$ is the total number of copes of $x$ in the boundary.



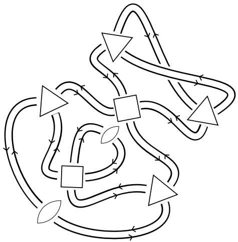

$\chi(S) = -2$
$-\chi(S)/2n(S) = 1$



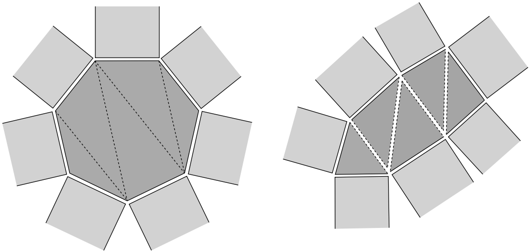

$\chi(S) = -4$
$-\chi(S)/2n(S) = \frac{2}{3}$

# Review (Building efficient surfaces)

If we have a labeled fatgraph, we can chop it into rectangles and polygons:

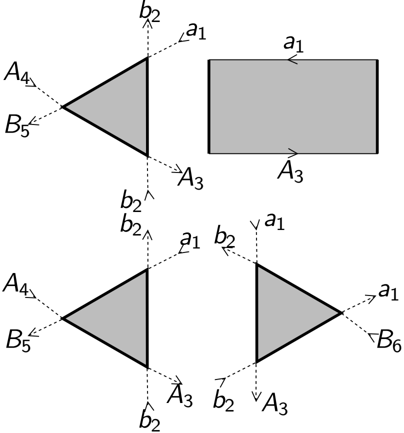


and we can chop the polygons into triangles:



# Review (Building efficient surfaces)

For a given boundary, there are only finitely many types of rectangles and triangles (pieces) which can appear.




If we take *any* collection of these pieces such that every edge appears the same number of times as its mate, then *any* way in which we glue them up will produce a fatgraph with the desired boundary.

# Review (Building efficient surfaces)

Given a boundary $x$, suppose there are $N$ pieces. A vector $X \in \mathbb{R}^N$ records how many of each piece there are. We define vectors:

$$(E_k)_i = \begin{cases} 1 & \text{if edge } e_k \text{ is in piece } i \\ -1 & \text{if edge } -e_k \text{ is in piece } i \\ 0 & \text{if neither edge is in piece } i \end{cases}$$

So if $X \cdot E_k = 0$ for all edge pairs $k$, the pieces can be glued up.

$$H_i = \begin{cases} 1 & \text{if the first letter of } x \text{ is in piece } i \\ 0 & \text{otherwise} \end{cases}$$

$$\chi_i = \begin{cases} \frac{-1}{2} & \text{if piece } i \text{ is a triangle} \\ 0 & \text{otherwise} \end{cases}$$

So when the pieces in $X$ are glued up to a fatgraph $S$, we'll have $n(S) = X \cdot H$, and we'll have $\chi(S) = X \cdot \chi$.

# Review (Building efficient surfaces)

So essentially, vectors $X \in \mathbb{R}^N$ correspond to labeled fatgraphs, so if we solve:

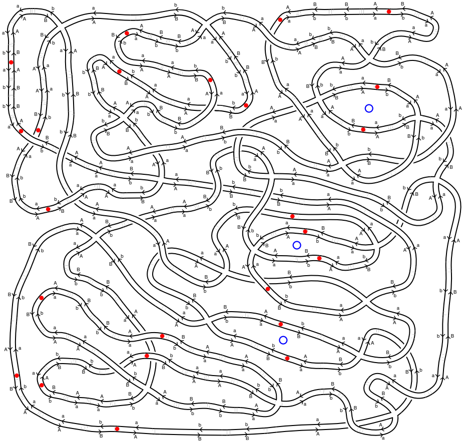
Find $X$ to minimize $-(X \cdot \chi)/2$, subject to the restriction that $X_i \geq 0$ and the matrix equation:

$$\begin{bmatrix} \text{---} & E_1 & \text{---} \\ \text{---} & E_2 & \text{---} \\ \text{---} & E_3 & \text{---} \\ & \vdots & \\ \text{---} & E_K & \text{---} \\ \text{---} & H & \text{---} \end{bmatrix} \begin{bmatrix} | \\ X \\ | \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Then this minimizing vector $X$ can be glued up into a most-efficient surface.

# Review (Linear programming)

Minimizing a linear function subject to linear inequalities is called *linear programming*, and there exist good algorithms for it (perhaps to be discussed later). Therefore, even though we started with an optimization over infinitely many surface maps, we have made the problem finite and tractable, even for large words:



# Efficient surfaces and scl

**Theorem (Calegari)**

*For any* $x \in [F, F]$,

$$\mathrm{scl}(x) = \inf_S \frac{-\chi(S)}{2n(S)}$$

Since surfaces with one boundary correspond to products of commutators, this theorem would be obvious if we restricted our surfaces to have one boundary component.

However, since the right side is an inf over all admissible surfaces (with arbitrarily many boundaries), the proof is harder.

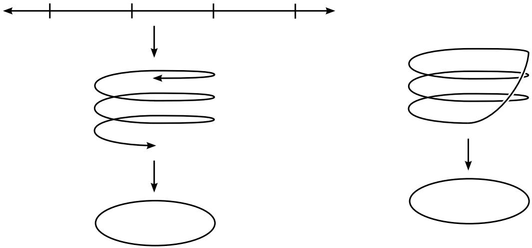
# Covers

A surface $S'$ *covers* another surface $S$ if there is a map $S' \to \S$ which is a local homeomorphism. (The surfaces are locally the same, but globally, $S'$ is bigger and wraps several times around $S$).

The number of preimages of a point is the *degree* of the cover.

## Example (1-dimensional example)

$\mathbb{R}$ covers $S^1$ with infinite degree. Also, $S^1$ covers itself with any desired degree.

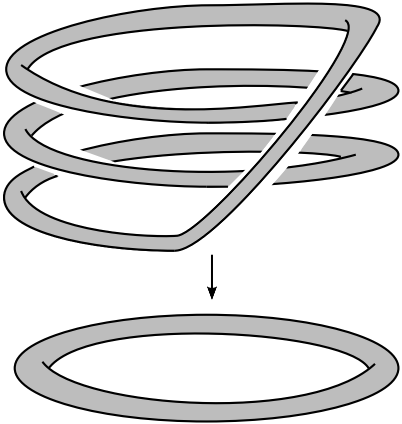


Covers of $S^1$ by $\mathbb{R}$ and $S^1$. The cover on the right is of degree 3.

# Covers

## Example

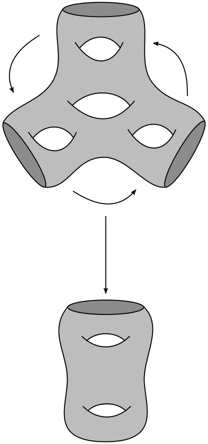
Just like $S^1$, a torus can cover itself with arbitrary degree.



# Covers

## Example

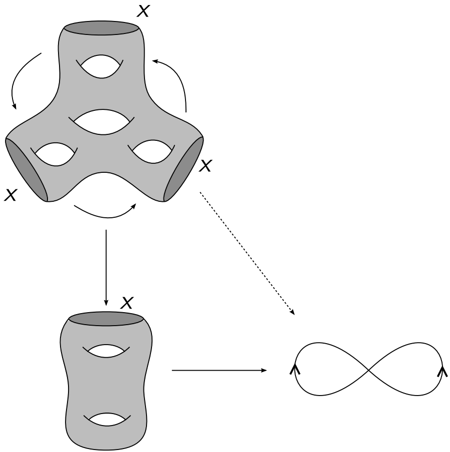
Surfaces with boundary can cover other surfaces with boundary. Usually the best way to imagine a covering map is to quotient a surface by a symmetry.



# Covers

## Example

If a surface $S'$ covers another surface $S$ with degree $d$, and there is a map from $S$ to the free group, then there is an induced map from the cover $S'$, and $n(S') = dn(S)$.



# Covers

**Lemma (Lemma 1)**

*If $S'$ covers $S$ with degree $d$, then $\chi(S') = d\chi(S)$.*

**Proof.**

Triangulations. □

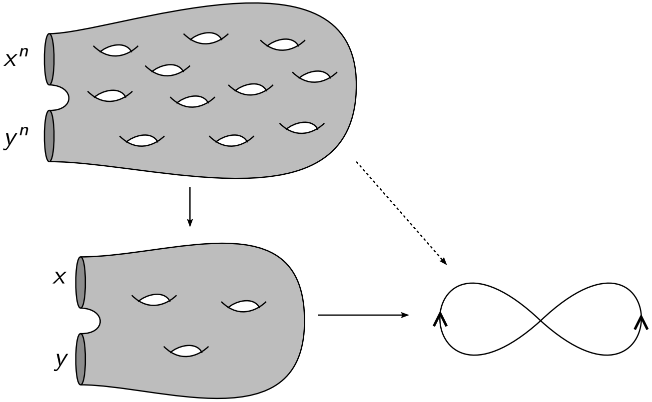
**Lemma (Lemma 2)**

*Suppose there is a map from $S$ to $F$ with $b$ boundaries. Then for any $N$, there is a cover $S'$ of $S$ of degree $d > N$ such that $S'$ also has $b$ boundaries.*

**Proof.**

Algebraic topology (Math 263). □

# Lemma 2 example




Lemma 2 says that given a surface map into $F$, we can find a find a cover of arbitrarily high degree $n$ with the same number of boundary components. The boundary components will become huge — they will cover the boundary components downstairs as $S^1$ covers $S^1$ of degree $n$, but there will be the same number of boundaries.

# Efficient surfaces and scl

### Theorem (Calegari)

*For any* $x \in [F, F]$, $\mathrm{scl}(x) = \inf_S \frac{-\chi(S)}{2n(S)}$.

### Proof part 1.

Recall $\mathrm{scl}(x) = \lim_{n\to\infty} \frac{\mathrm{cl}(x^n)}{n}$. We'll prove the equality by proving two inequalities.

Proof that $\inf_S \frac{-\chi(S)}{2n(S)} \leq \mathrm{scl}(x)$: Let $m$ be such that $\frac{\mathrm{cl}(x^m)}{m} < \mathrm{scl}(x) + \frac{\epsilon}{2}$ and $\frac{1}{2m} < \frac{\epsilon}{2}$. Set $k = \mathrm{cl}(x^m)$, so $x^m$ can be expressed as a product of $k$ commutators, which means there is a surface map $S'$ of genus $k$ with one boundary $x^m$. Then $\chi(S') = 1 - 2k$ and $n(S') = m$, so

$$\inf_S \frac{-\chi(S)}{2n(S)} \leq \frac{-\chi(S')}{2n(S')} = \frac{2k-1}{2m} = \frac{k}{m} - \frac{1}{2m} < \mathrm{scl}(x) + \epsilon$$

This inequality holds for all $\epsilon > 0$, so we are done this direction.

□

# Efficient surfaces and scl

Theorem (Calegari)

*For any* $x \in [F, F]$, $\text{scl}(x) = \inf_S \frac{-\chi(S)}{2n(S)}$.

Proof part 2:

Other equality (proof that $\text{scl}(x) \leq \inf_S \frac{-\chi(S)}{2n(S)}$): Let $S'$ be a surface with $\frac{-\chi(S')}{2n(S')} < \inf_S \frac{-\chi(S)}{2n(S)} + \frac{\epsilon}{2}$.

Let $n = n(S')$, and suppose that $S'$ has $k$ boundaries.

There is a cover $T$ of $S$ with $k$ boundaries and degree $d$ such that $\frac{k}{2nd} < \frac{\epsilon}{2}$. Tube the boundaries together to get a surface $T'$ with a single boundary $x^{dn}$. We compute
$\chi(T') = \chi(T) - (k-1) = d\chi(S') - (k-1)$.

# Efficient surfaces and scl

The genus of $T'$ is $\frac{1-\chi(T')}{2}$, so there is an expression for $x^{dn}$ as a product of $\frac{1-\chi(T')}{2}$ commutators. Therefore,

$$\text{scl}(x) \leq \frac{\text{cl}(x^{dn})}{dn} \leq \frac{\frac{1-\chi(T')}{2}}{dn}$$

and

$$\frac{1-\chi(T')}{2dn} = \frac{1-d\chi(S')+(k-1)}{2dn} = \frac{-\chi(S')}{2n} + \frac{k}{2dn}$$

and

$$\frac{-\chi(S')}{2n} + \frac{k}{2dn} \leq \inf_S \frac{-\chi(S)}{2n(S)} + \epsilon$$

This holds for any $\epsilon > 0$, so this direction is done.

# Efficient surfaces and scl

### Theorem (Calegari)

*For any* $x \in [F, F]$, $\mathrm{scl}(x) = \inf_S \frac{-\chi(S)}{2n(S)}$.

Therefore, we can compute scl by computing a most efficient surface with boundary $x$, using the algorithm from last time.

This yields surprising complexity.

# Experimental histogram

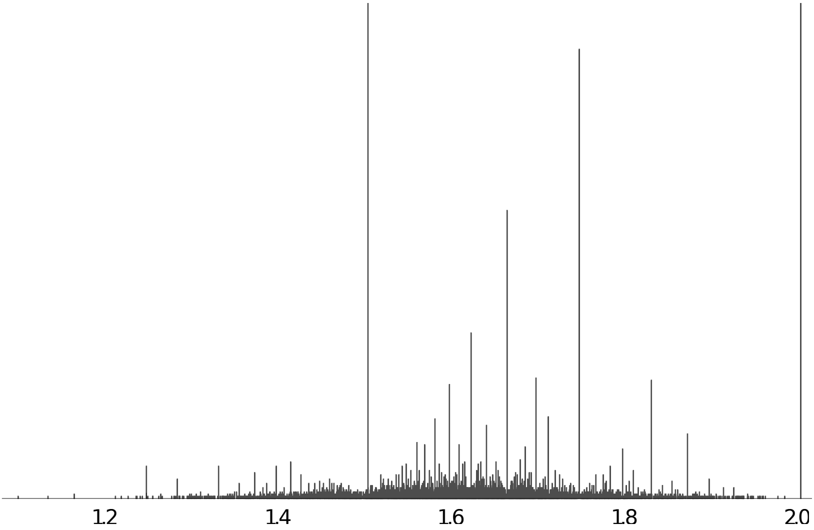
We get a *random word* of length $n$ in $F$ by picking a word uniformly at random from all words of length $n$ in $F$.

We get a random word of length $n$ in $[F, F]$ by restricting our choice to $[F, F]$.

What happens if we pick lots of random words in $[F, F]$ of length 40 and compute scl?

# Random loops

We get this histogram of observed frequencies of scl values.



# Groups

A *group* is a pair $(G, \cdot)$, where $G$ is a set, and $\cdot : G \times G \to G$ is a binary operation such that:

1. For all $a, b, c \in G$, we have $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
2. There is $e \in G$ (called the *identity*) such that $eg = ge = g$ for all $g \in G$.
3. For all $a \in G$ there is $b \in G$ (usually written $a^{-1}$) such that $a \cdot b = b \cdot a = e$.

# Groups

Examples of groups:

- A free group $F$ with the product operation
- $\mathbb{R}$ under addition ($e = 0$)
- $\mathbb{Z}$ under addition ($e = 0$)
- Any vector space under addition ($e = 0$)
- The symmetries of an object e.g, the group $D_n$ (also known as $D_{2n}$), the symmetries of a regular $n$-gon.

# Homomorphisms

A map $f : G \to H$ between groups $G$ and $H$ is a *homomorphism* if $f(g \cdot h) = f(g) \cdot f(h)$. Notice that it must be that $f(e_G) = e_H$, since $f(g) = f(e_G \cdot g) = f(e_G) \cdot f(g)$, so $e_H = f(g) \cdot (f(g))^{-1} = f(e_G)$.

Examples:

- The map $f : \mathbb{Z} \to \mathbb{Z}$ defined $f(x) = 2x$.
- The map $f : F \to \mathbb{Z}$ defined by $f(w) =$ (signed number of $a$'s)
- Any linear map between vector spaces

# Quasimorphisms

A function $f : G \to \mathbb{R}$ is a *quasimorphism* if

$$\sup_{g,h \in G} |f(g) + f(h) - f(g \cdot h)| < \infty$$

We define the *defect*

$$D(f) = \sup_{g,h \in G} |f(g) + f(h) - f(g \cdot h)|$$

Notice that if $D(f) = 0$, then $f$ is a homomorphism. We say that $f$ is *homogenous* if

$$f(g^n) = nf(g)$$

For all $g \in G$ and $n \in \$Z$.

### Theorem (Bavard)

*Let $Q$ be the set of all quasimorphisms $F \to \mathbb{R}$. For $x \in [F, F]$, we have*

$$\operatorname{scl}(x) = \sup_{f \in Q} \frac{f(x)}{2D(f)}$$

# Surfaces and commutators (Geometry REU) Class 5

Alden Walker
(Later: Danny Calegari)

Summer 2013

# Review (Efficient surfaces and scl)

### Theorem (Calegari)

*For any* $x \in [F, F]$, $\mathrm{scl}(x) = \inf_S \frac{-\chi(S)}{2n(S)}$.

### Proof.

Given an expression for $x^n$ in commutators, we get a surface map, so $\inf_S \frac{-\chi(S)}{2n(S)} \leq \mathrm{scl}(x)$.

Conversely, given a surface map, we can take a huge cover such that connecting the boundaries to make it have a single boundary affects $\chi$ a tiny amount, so we get an expression for $x^n$ as a product of commutators, so $\mathrm{scl}(x) \leq \inf_S \frac{-\chi(S)}{2n(S)}$. □

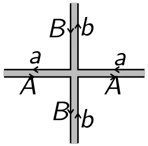
# cl($[a, b]^n$)

## Lemma (Culler)

$\mathrm{cl}([a, b]^n]) = \left\lfloor \frac{n}{2} \right\rfloor + 1$

**Proof:**

Every polygon in a fatgraph with boundary $[a, b]^n$ must have valence at least 4. This is because the order of the letters means polygons simply can't close up until we see at least four rectangles.




The smallest magnitude Euler characteristic is achieved when there are as many vertices as possible, i.e. when every vertex has valence 4, so

$$-\chi(S) \geq V - V/2 = 2n - n = n$$

for a surface with boundary $[a, b]^n$.

# $\mathrm{cl}([a,b]^n)$

## Lemma (Culler)

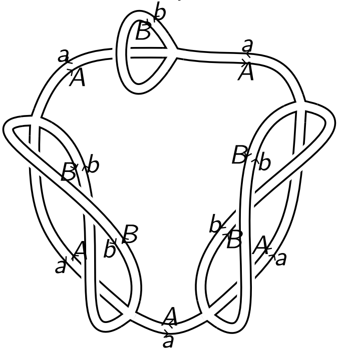
$\mathrm{cl}([a,b]^n]) = \left\lfloor \frac{n}{2} \right\rfloor + 1$

**Proof continued:**

Therefore, for the genus $g$ of a surface with boundary $[a,b]^n$, we have

$$g \geq \frac{1 + (-\chi)}{2} = \frac{n+1}{2}$$

If $n$ is odd, we can construct an explicit surface with $g = \frac{n+1}{2}$



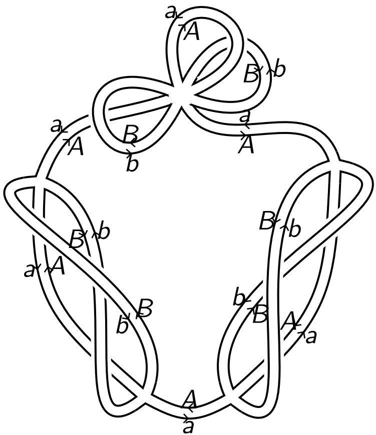
# $\mathrm{cl}([a, b]^n)$

## Lemma (Culler)

$\mathrm{cl}([a, b]^n]) = \lfloor \frac{n}{2} \rfloor + 1$

**Proof continued:**

If $n$ is even, note $g$ must be an integer, so $g \geq \frac{n}{2} + 1$. We just add one genus to realize $g = \frac{n}{2} + 1$.



# Stable $W$-length

Reference for this section: Calegari, Danny and Zhuang, Dongping, *Stable $W$-length*; arXiv:1008.2219

# Stable $W$-length

Let $W$ be some word in a free group $F_n$ of arbitrary rank $n$ (a word in the letter $x_1, x_2, \dots, x_n$). A *$W$-word* in $F$ is a word in $F$ of the form $f(W)$ for some homomorphism $f : F_n \to F$. This is a word in the "pattern" of $W$, replacing $x_1, \dots, x_n$ with arbitrary words in $F$.

### Example

Set $W = [x_1, x_2]$. Then the set of $W$-words is all commutators.

### Example

Set $W = [x_1, [x_2, x_3]]$. Then $[a, b]$ is *not* a $W$-word, but $[a, [a, b]] = aabABAbaBA$ *is* a $W$-word.

### Example

Set $W = [[x_1, [x_2, x_1]]$. Then $[a, [a, b]]$ is not a $W$-word, but $[ab, [aba, ab]]$ is.

# Stable $W$-length

The *verbal subgroup* $F_W$ of $F$ is the subgroup of $F$ generated by all $W$-words. $F_W$ is the set of all products of $W$-words.

**Example**

$[F, F]$ is the verbal subgroup for the word $[x_1, x_2]$.

**Example**

The word $[a, [a, b]][ab, [ba, ab]]$ is in the verbal subgroup for the word $[x_1, [x_2, x_3]]$.

Given $x \in F_W$, we define the *$W$-length* $\ell(x|W)$ of $x$ to be the smallest number of $W$-words whose product is $x$.

**Example**

For $W = [x_1, x_2]$, $\ell(x|W) = \mathrm{cl}(x)$.

# Stable $W$-length

We define

$$\mathrm{sl}(x|W) = \lim_{n\to\infty} \frac{\ell(x^n|W)}{n}$$

So note $\mathrm{scl}(x) = \mathrm{sl}(x|[x_1, x_2])$.

In general, $\mathrm{sl}(x, W)$ is very mysterious.

Example

If $w \in F$, then we can ask for $\ell(w|w)$ and $\mathrm{sl}(w|w)$. E.g. $\mathrm{scl}([a, b]) = \mathrm{sl}([a, b]|[a, b]) = \frac{1}{2}$.

# Stable $W$-length

## Lemma

*Suppose that* $w \notin [F, F]$ *(i.e. $w$ is not a product of commutators). then* $\mathrm{sl}(g|w) = 0$.

## Proof.

If $w \notin [F, F]$, then (wlog) $a$ appears $m$ more times than $A$. Given $g \in F$, substitute $a \to g$ and $b \to e$; this gives $g^m$ as a $w$-word. This holds for *all* $g \in F$, so

$$\mathrm{sl}(g|w) = \lim_{n\to\infty} \frac{\ell(g^n|w)}{n} = \lim_{n\to\infty} \frac{\ell(g^{mn}|w)}{mn} = \lim_{n\to\infty} \frac{1}{mn} = 0$$

□

# Stable $W$-length

**Lemma (Calegari-Zhuang)**

*For $w \in F$,*

$$\frac{1}{2} \leq \frac{\mathrm{scl}(w)}{\mathrm{scl}(w)+1} \leq \mathrm{sl}(w|w) \leq 1$$

**Lemma (Calegari-Zhuang)**

*For any $n$,*

$$\mathrm{sl}(g|w) \leq \frac{\ell(g^n|w) - 1 + \mathrm{sl}(w|w)}{n}$$

# Stable $W$-length

## Lemma (Calegari-Zhuang)

*For any $n$,* $\mathrm{sl}(g|w) \leq \frac{\ell(g^n|w)-1+\mathrm{sl}(w|w)}{n}$

Proof:

Set $m = \ell(g^n|w)$. We can write: $g^n = w_1 \cdots w_m$, where the $w_i$ are $w$-words. So for any $k$,

$$g^{nk} = (w_1 \cdots w_m)(w_1 \cdots w_m) \cdots (w_1 \cdots w_m)$$

Note $w_m w_1 = w_1(w_1^{-1} w_m w_1)$, and $w_1^{-1} w_m w_1$ is a $w$-word (exercise!), so

$$g^{nk} = w_1 \cdots w_{m-1} w_1 w_m' w_2 \cdots w_m \cdots (w_1 \cdots w_m)$$

I.e. we can move the copies of $w_1$ around freely in the word.

# Stable $W$-length

Lemma (Calegari-Zhuang)

*For any $n$,* $\text{sl}(g|w) \leq \frac{\ell(g^n|w)-1+\text{sl}(w|w)}{n}$

Proof part 2:

So

$$g^{nk} = w_1^k h_1 h_2 \cdots h_{(m-1)k}$$

where the $h_i$ are $w$-words. For $k$ large enough, we can write $w_1^k$ as a product of $k\text{sl}(w|w) + o(k)$ $w$-words, so $g^{nk}$ is a product of $(m-1)k + k\text{sl}(w|w) + o(k)$ $w$-words, so

$$\text{sl}(g|w) \leq \frac{(m-1)k + k\text{sl}(w|w) + o(k)}{nk}$$

Take $k \to \infty$ and plug in $m = \ell(g|w)$.

# Stable $W$-length

## Lemma (Calegari-Zhuang)

*For any $w \in F$, if $\ell(w^n|w) = m$, then*

$$\mathrm{sl}(w|w) \leq \frac{m-1}{n-1}$$

## Proof.

Plug in $g = w$ to the previous lemma. □

# Stable $W$-length

Example (Calegari-Zhuang)

$\frac{2}{3} \leq \mathrm{sl}([a,b]^2 | [a,b]^2) \leq \frac{4}{5}$.

Proof:

By Culler's equality ($[a,b]^3 = [aba^{-1}, b^{-1}aba^{-2}][b^{-1}ab, bb]$), if $x = [aba^{-1}, b^{-1}aba^{-2}]$ and $y = [b^{-1}ab, bb]$, then

$$([a,b]^2)^6 = ([a,b]^3)^4 = xyxyxy = xxyyzxxyyz$$

for $z = [y^{-1}, y^{-1}x^{-1}y^{-1}]$, and there are commutators $x'$, $y'$ so that

$$xxyyzxxyyz = x^2y^2z^2(x')^2(y')^2$$

Therefore, $([a,b]^2)^6$ is a product of 5 squares-of-commutators ($w$-words).

# Stable $W$-length

$\frac{2}{3} \leq \mathrm{sl}([a,b]^2 | [a,b]^2) \leq \frac{4}{5}$.

Proof part 2:

Therefore, $([a,b]^2)^6$ is a product of 5 squares-of-commutators ($w$-words). By the lemmas (with $m = 6$, $n = 5$), then,

$$\frac{2}{3} = \frac{\mathrm{scl}([a,b]^2)}{\mathrm{scl}([a,b]^2) + 1} \leq \mathrm{sl}([a,b]^2 | [a,b]^2) \leq \frac{m-1}{n-1} = \frac{4}{5}$$

Back to quasimorphisms.

# Review (Groups)

A *group* is a pair $(G, \cdot)$, where $G$ is a set, and $\cdot : G \times G \to G$ is a binary operation such that:

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# Review (homomorphisms)

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Examples:

- The map $f : \mathbb{Z} \to \mathbb{Z}$ defined $f(x) = 2x$.
- The map $f : F \to \mathbb{Z}$ defined by $f(w) =$ (signed number of $a$'s)
- Any linear map between vector spaces

# Review (quasimorphisms)

A function $f : G \to \mathbb{R}$ is a *quasimorphism* if

$$\sup_{g,h \in G} |f(g) + f(h) - f(g \cdot h)| < \infty$$

We define the *defect*

$$D(f) = \sup_{g,h \in G} |f(g) + f(h) - f(g \cdot h)|$$

Notice that if $D(f) = 0$, then $f$ is a homomorphism. We say that $f$ is *homogenous* if

$$f(g^n) = nf(g)$$

For all $g \in G$ and $n \in \mathbb{Z}$.

# Review (Duality)

### Theorem (Bavard)

*Let $Q$ be the set of all homogeneous quasimorphisms $F \to \mathbb{R}$. For $x \in [F, F]$, we have*

$$\mathrm{scl}(x) = \sup_{f \in Q} \frac{f(x)}{2D(f)}$$

# Quasimorphism example

Let $\sigma \in F$. Define $C_\sigma : F \to \mathbb{Z}$ and $c_\sigma : F \to \mathbb{Z}$ by

$$C_\sigma(w) = \text{number of copies of } \sigma \text{ in } w$$

$$c_\sigma(w) = \text{max number of non-overlapping copies of } \sigma \text{ in } w$$

Example

$C_a(aba) = 2$, $C_{aba}(ababa) = 2$, $c_{aba}(ababa) = 1$.

# Quasimorphism example

Define $H_\sigma, h_\sigma : F \to \mathbb{R}$ by

$$H_\sigma(w) = C_\sigma(w) - C_\sigma(w^{-1}) = C_\sigma(w) - C_{\sigma^{-1}}(w)$$

$$h_\sigma(w) = c_\sigma(w) - c_\sigma(w^{-1}) = c_\sigma(w) - c_{\sigma^{-1}}(w)$$

There are the *big* and *small* counting quasimorphisms.

Example

$H_{ab}(abABAbaB) = 1 - 1 = 0$

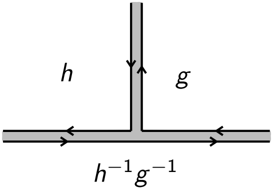
Claim: for all $\sigma \in F$, $H_\sigma$ and $h_\sigma$ are quasimorphisms. Note they are *antisymmetric*: $H_\sigma(w^{-1}) = -H_\sigma(w)$.

# Defect

In general, it is very difficult to find $D(f)$ for a quasimorphism. However, there is a graphical way to understand defect, and for $H_\sigma$ and $h_\sigma$, this succeds in computing it.

# Tripods

Recall $D(f) = \sup_{g,h} |f(g) + f(h) - f(gh)|$. When we write $gh$, some of the letters in the middle will cancel.



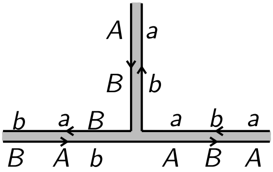

If $f$ is antisymmetric, then graphically, the defect expression is the value of the quasimorphism on the boundary of the tripod, because the bottom edge reads off $h^{-1}g^{-1} = (gh)^{-1}$, so

$$f(g) + f(h) + f((gh)^{-1}) = f(g) + f(h) - f(gh)$$

So

$$D(f) = \sup_T |f(\partial T)|$$

# Tripod example



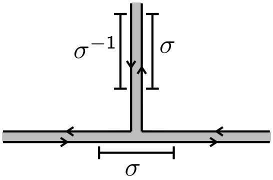

Graphically, this tripod $T$ shows that $((ababa)(ABBab))^{-1} = (abaBab)^{-1} = BAbABA$. So if $f$ is a antisymmetric quasimorphism, then

$$\begin{aligned} f(\partial T) &= f(ababa) + f(ABBab) + f(BAbABA) \\ &= f(ababa) + f(ABBab) + f(((ababa)(ABBab))^{-1}) \\ &= f(ababa) + f(ABBab) - f((ababa)(ABBab)) \end{aligned}$$

The supremum of $f(\partial T)$ over all tripods is therefore $D(f)$.

# Tripods (Defect of $H_\sigma$)

Consider applying $H_\sigma$ to $\partial T$ for some tripod $T$. If an occurrence of $\sigma$ does not cross the junction, then there will be a matching copy of $\sigma^{-1}$ on the other side, so there will be no contribution to $H_\sigma(\partial T)$.




The top right copy of $\sigma$ does not cross the junction, and is cancelled out. The bottom copy of $\sigma$ is not cancelled and does contribute to $H_\sigma(\partial T)$.

Therefore, $D(H_\sigma) = \max_T H_\sigma(\partial T)$, where the maximum is taken over the *finite* set of tripods with arm length less than $|\sigma|$.

# Tripods (Defect of $h_\sigma$)

Again, $D(h_\sigma) = \sup_T h_\sigma(\partial T)$. However, interactions between copies of $\sigma$ can interact, making computation hard.

## Example

$h_{aba}(abababa) = 2$.

Here it's not clear that we can limit to finitely many tripods. However, we can actually get a *uniform* defect bound

## Lemma

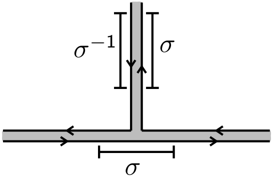
*For any $\sigma$, $D(h_\sigma) \leq 3$*

# Defect of $h_\sigma$

## Lemma

*For any $\sigma$, $D(h_\sigma) \leq 3$*

Proof: First compute $c_\sigma(\partial T)$. Consider the copies of $\sigma$ which contribute. Not that for all the copies that do not overlap the junction, there is a matching copy of $\sigma^{-1}$ on the other side. All these copies of $\sigma^{-1}$ are disjoint.




When we compute $c_{\sigma^{-1}}(\partial T)$, it is possible that we can do better – i.e. there is some configuration with more copies of $\sigma^{-1}$. However, we do get that

$$c_{\sigma^{-1}}(\partial T) \geq c_\sigma(\partial T) - 3$$

(There are at most 3 copies of $\sigma$ overlapping the junction)

# Defect of $h_\sigma$

Proof part 2: We just saw

$$c_{\sigma^{-1}}(\partial T) \geq c_\sigma(\partial T) - 3$$

An identical argument swapping $\sigma^{-1}$ for $\sigma$ gives

$$c_\sigma(\partial T) \geq c_{\sigma^{-1}}(\partial T) - 3$$

and therefore,

$$3 \geq c_\sigma(\partial T) - c_{\sigma^{-1}}(\partial T) \geq -3$$

I.e. $D(h_\sigma) \leq 3$.

# Homogenization

Given any quasimorphism $f$, we define the homogenization $\overline{f}$ by

$$\overline{f}(x) = \lim_{n\to\infty} \frac{f(x^n)}{n}$$

Note that the homogenization is homogenous.

## Lemma

$D(f) \leq D(\overline{f}) \leq 2D(f)$

# Homogenization of $H_\sigma$

Note

$$\overline{H}_\sigma(w) = \lim_{n\to\infty} \frac{H_\sigma(w^n)}{n}$$

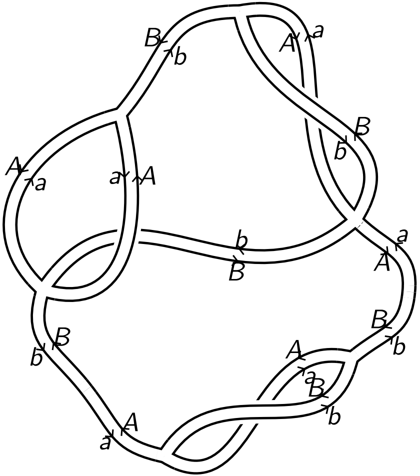
This seems hard to compute, but actually, $\overline{H}_\sigma(w)$ is the (signed) number of copies of $w$ in the *cyclic* word $w$.

Example

$\overline{H}_a(aba) = 2$, $\overline{H}_{aba}(abab) = 2$. There is a copy of *aba* starting on the third letter of *abab*.

# Relationship to surfaces

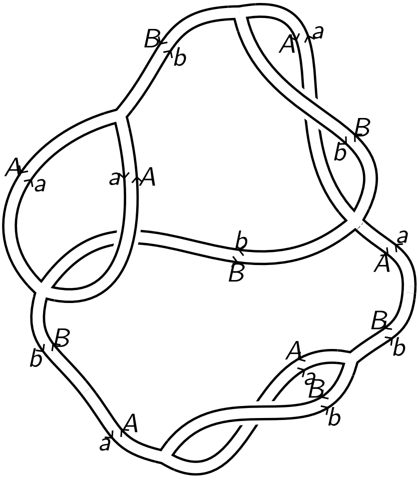
Why should $\sup_f \frac{f(x)}{2D(f)}$ be related to surfaces and $\mathrm{scl}(x)$?




This is an extremal surface for $aBAbaaBABAbb$. We have $-\chi(S) = -4$ and $n(S) = 2$, so $\mathrm{scl} = \frac{4}{2 \cdot 2} = 1$.

# Relationship to surfaces

To apply the quasimorphism $\overline{H}_{BA} + \overline{H}_{aB} + \overline{H}_{Ab} + \overline{H}_{ba}$ to the word *aBAbaaBABAbb*, we can add up the values centered on each vertex. Let's do the top one.




We get the words $aB + bb + BA$, so it contributes $1 + 0 + 1 = 2$.

# Applying a quasimorphism to a surface

Here is a table of the values we get around each of the vertices, with the valence of the vertex recorded:

| Vertex | Valence | Value of $f = \overline{H}_{BA} + \overline{H}_{aB} + \overline{H}_{Ab} + \overline{H}_{ba}$ |
|---|---|---|
| top | 3 | 2 |
| top left | 3 | 2 |
| top right | 4 | 4 |
| middle left | 4 | 4 |
| bottom left | 3 | 2 |
| bottom right | 3 | 2 |

So the total value of $f$ on $2aBAbaaBABAbb$ is 16. It's a fact that $D(f) = 4$. Notice:

- $f$ takes $\frac{1}{2}$ its defect on all tripods embedded in the surface (a vertex of valence 4 is two tripods stuck together).
- $f$ is extremal: $\frac{f(aBAbaaBABAbb)}{2D(f)} = 1 = \mathrm{scl}(aBAbaaBABAbb)$.

This is not an accident!

# Quasimorphisms and surfaces

### Lemma

*Suppose that $f$ is a homogenized counting quasimorphism of length 2 Then $f$ is extremal for $x$ ($\frac{f(x)}{2D(f)} = \mathrm{scl}(x)$) if and only if $f$ takes $\frac{1}{2}$ its defect on every tripod in every extremal surface for $x$.*

### Proof.

Suppose there are $M$ tripods in an extremal surface $S$. Then $\chi = -M/2$. Therefore, $\mathrm{scl}(x) = M/4n(S)$. Scale $f$ so $D(f) = 1$. So $f(n(S)x) = M/2$ (takes $\frac{1}{2}$ on all tripods) if and only if $f(x)/2 = M/4n(S) = \mathrm{scl}(x)$. □