Surfaces and commutators
(Geometry REU)
Class 6

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Review (quasimorphisms)

A function $f : G \to \mathbb{R}$ is a *quasimorphism* if

$$\sup_{g,h \in G} |f(g) + f(h) - f(g \cdot h)| < \infty$$

We define the *defect*

$$D(f) = \sup_{g,h \in G} |f(g) + f(h) - f(g \cdot h)|$$

Notice that if $D(f) = 0$, then f is a homomorphism. We say that f is *homogenous* if

$$f(g^n) = nf(g)$$

For all $g \in G$ and $n \in \mathbb{Z}$.

Review (Duality)

Theorem (Bavard)

Let Q be the set of all homogeneous quasimorphisms $F \to \mathbb{R}$. For $x \in [F, F]$, we have

$$\mathrm{scl}(x) = \sup_{f \in Q} \frac{f(x)}{2D(f)}$$

Review (Counting quasimorphisms)

Let $\sigma \in F$. Define $C_\sigma : F \to \mathbb{Z}$ and $c_\sigma : F \to \mathbb{Z}$ by

$$C_\sigma(w) = \text{number of copies of } \sigma \text{ in } w$$

$$c_\sigma(w) = \text{max number of non-overlapping copies of } \sigma \text{ in } w$$

Define $H_\sigma, h_\sigma : F \to \mathbb{R}$ by

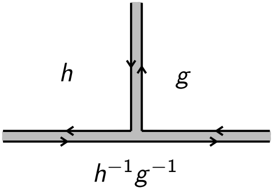
$$H_\sigma(w) = C_\sigma(w) - C_\sigma(w^{-1}) = C_\sigma(w) - C_{\sigma^{-1}}(w)$$

$$h_\sigma(w) = c_\sigma(w) - c_\sigma(w^{-1}) = c_\sigma(w) - c_{\sigma^{-1}}(w)$$

These are the *big* and *small* counting quasimorphisms.

Review (Tripods)

Recall $D(f) = \sup_{g,h} |f(g) + f(h) - f(gh)|$. When we write gh, some of the letters in the middle will cancel.



If f is antisymmetric, then graphically, the defect expression is the value of the quasimorphism on the boundary of the tripod, because the bottom edge reads off $h^{-1}g^{-1} = (gh)^{-1}$, so

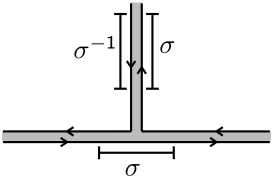
$$f(g) + f(h) + f((gh)^{-1}) = f(g) + f(h) - f(gh)$$

So

$$D(f) = \sup_T |f(\partial T)|$$

Review (Computing the defect of H_σ)

Consider applying H_σ to ∂T for some tripod T. If an occurrence of σ does not cross the junction, then there will be a matching copy of σ^{-1} on the other side, so there will be no contribution to $H_\sigma(\partial T)$.



The top right copy of σ does not cross the junction, and is cancelled out. The bottom copy of σ is not cancelled and does contribute to $H_\sigma(\partial T)$.

Therefore, $D(H_\sigma) = \max_T H_\sigma(\partial T)$, where the maximum is taken over the *finite* set of tripods with arm length less than $|\sigma|$.

Review (Defect of h_σ)

Lemma

For any σ, $D(h_\sigma) \leq 3$

Proof.

This is harder, since h_σ counts the max over disjoint words, so the interactions are complicated. However, for every copy of σ that counts towards c_σ and does not overlap the boundary, we get at least one copy of σ^{-1} counting towards $c_{\sigma^{-1}}$. Therefore,

$$c_{\sigma^{-1}}(\partial T) \geq c_\sigma(\partial T) - 3$$

We get a similar bound swapping σ and σ^{-1}. □

Review (Homogenization)

Given any quasimorphism f, we define the homogenization $\overline{f}$ by

$$\overline{f}(x) = \lim_{n\to\infty} \frac{f(x^n)}{n}$$

Note that the homogenization is homogenous.

Lemma

$D(f) \leq D(\overline{f}) \leq 2D(f)$

Lemma

For any σ, $D(\overline{H_\sigma}) = 2D(H_\sigma)$.

Facts about homogeneous quasimorphisms

Lemma

To apply $\overline{H_\sigma}$ to a word w, we think of w as a cyclic word, and we count subwords, e.g. $H_{ab}(bAba) = 1$.

In general,

Lemma

A homogeneous quasimorphism has a well-defined value on a cyclic word.

Proof.

We use square brackets to denote a cyclic word, like $w = [w_1 \cdots w_n]$, and $w_2 \cdots w_n w_1$ is the regular word we get by starting at the second letter of w and reading around.
If f is a homogeneous quasimorphism, then it's a fact (exercise) that

$$f(w_i \cdots w_n w_1 \cdots w_{i-1}) = f(w_j \cdots w_n w_1 \cdots w_{j-1})$$

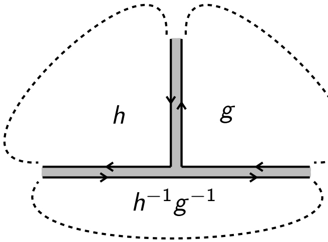
for all i and j (the value doesn't depend on where we start). □

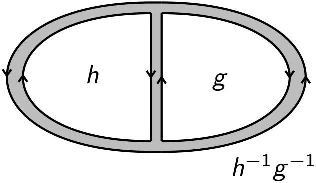
Defects of homogeneous quasimorphisms

We saw that for an antisymmetric quasimorphism f, $D(f) = \sup_T f(\partial T)$.

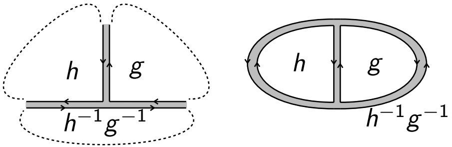
When f is a big counting quasimorphism H_σ, this reduces to a finite set of tripods, and we can compute $D(H_\sigma)$, and we know $D(\overline{H_\sigma}) = 2D(H_\sigma)$.

When f is homogeneous (e.g. $\overline{H_\sigma}$), it's still true that $D(f) = \sup_T f(\partial T)$. However, since homogeneous quasimorphisms can be applied to cyclic words, we can also think of it as $D(f) = \sup_P f(\partial P)$, where P ranges over all thrice-punctured spheres.





Defects of homogeneous quasimorphisms



This picture is the proof of:

Lemma

$D(\overline{H_\sigma}) = 2D(H_\sigma)$

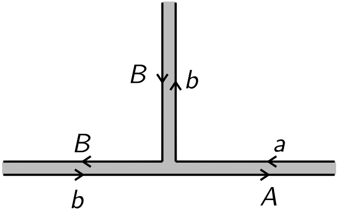
Proof.

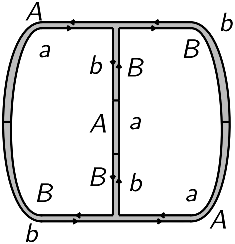
We know that there is a finite tripod T on which H_σ takes its defect. Take two copies of T at both junctions of a thrice-punctured sphere P; the value of $\overline{H_\sigma}$ on P is $2D(H_\sigma)$. This proves $D(\overline{H_\sigma}) \geq 2D(H_\sigma)$, and we know the reverse inequality for more complicated reasons. □

Defects of homogeneous quasimorphisms

Example

It's a fact that $D(H_{ab}) = 1$, which is realized by the tripod shown.





It's also a fact that $D(\overline{H_{ab}}) = 2$, realized by the thrice-punctured sphere shown, which we constructed by taking two copies of the tripod for H_{ab} (it doesn't matter what we fill in the middle, since all those words cancel).

Really, these are just graphical ways of recording that
$|H_{ab}(ab) + H_{ab}(BB) - H_{ab}(aB)| = 1$ and
$|\overline{H}_{ab}(BabaB) + \overline{H}_{ab}(bABBa) - \overline{H}_{ab}(BaBa)| = 2$, but thinking of them as bits of surfaces is usful.

Review (Connections to surfaces)

Recall

Theorem (Bavard)

Let Q be the set of all homogeneous quasimorphisms $F \to \mathbb{R}$. For $x \in [F, F]$, we have $\mathrm{scl}(x) = \sup_{f \in Q} \frac{f(x)}{2D(f)}$.

Why are quasimorphisms connected to surfaces? Because homogeneous quasimorphisms assign a uniformly bounded number to a thrice-punctured sphere, and surfaces are built out of thrice-punctured spheres.

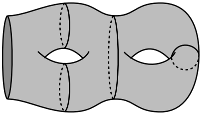
Connections to surfaces

Lemma

A homogeneous quasimorphism f is extremal for $x \in [F, F]$ (i.e. $\left(\frac{f(x)}{2D(f)} = \mathrm{scl}(x)\right)$ if and only if f takes its defect on all thrice-punctured spheres in all extremal surfaces for x.

Proof.

Given an extremal surface S for x, cut it into M thrice-punctured spheres:



Note that $\chi(S) = -M$, so $\mathrm{scl}(x) = \frac{M}{2n(S)}$. Then f takes its defect on all the thrice-punctured spheres iff $f(n(S)x) = MD(f)$, iff

$$\frac{f(x)}{2D(f)} = \frac{\frac{MD(f)}{n(S)}}{2D(f)} = \frac{M}{2n(S)} = \mathrm{scl}(x)$$

Quasimorphisms and surfaces

It's often easier to think about *un*homogenized counting quasimorphisms, in which case the following lemma is useful.

Lemma

Suppose that f is a (un)homogenized counting quasimorphism of length 2 Then $\overline{f}$ is extremal for x $\left(\frac{\overline{f}(x)}{2D(\overline{f})} = \mathrm{scl}(x)\right)$ if and only if f takes its defect on every tripod in every extremal surface for x.

Proof.

Suppose there are M tripods in an extremal surface S. Then $\chi = -M/2$. Therefore, $\mathrm{scl}(x) = M/4n(S)$.

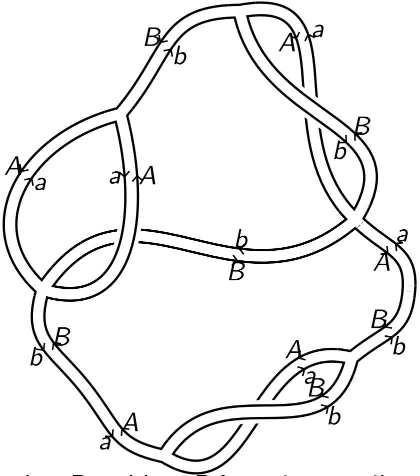
Now f takes its defect on all tripods iff $\overline{f}(n(S)x) = MD(f)$ iff

$$\frac{\overline{f}(x)}{2D(\overline{f})} = \frac{\overline{f}(x)}{4D(f)} = \frac{MD(f)}{4n(S)D(f)} = \frac{M}{4n(s)} = \mathrm{scl}(x).$$

□

Review (Example)

To apply the quasimorphism $\overline{H}_{BA} + \overline{H}_{aB} + \overline{H}_{Ab} + \overline{H}_{ba}$ to the word $aBAbaaBABAbb$, we can add up the values of $H_{BA} + H_{aB} + H_{Ab} + H_{ba}$ on the tripods of arm length 1 centered on each vertex. Let's do the top one.



We get the words $aB + bb + BA$, so it contributes $1 + 0 + 1 = 2$.

Review (Applying a quasimorphism to a surface)

Here is a table of the values we get around each of the vertices, with the valence of the vertex recorded:

Vertex	Valence	Value of $f = H_{BA} + H_{aB} + H_{Ab} + H_{ba}$
top	3	2
top left	3	2
top right	4	4
middle left	4	4
bottom left	3	2
bottom right	3	2

So the total value of f on $2aBAbaaBABAbb$ is 16. $D(f) = 2$, so $D(\overline{f}) = 4$. Notice:

- f takes its defect on all tripods embedded in the surface (a vertex of valence 4 is two tripods stuck together).
- $\overline{f}$ is extremal: $\frac{\overline{f}(aBAbaaBABAbb)}{2D(\overline{f})} = 1 = \text{scl}(aBAbaaBABAbb)$.

Finding quasimorphisms

Given an extremal surface for x, to find an extremal quasimorphism, we need to find a quasimorphism that takes its defect on all tripods.

So we are looking for a function on words, subject to constrained values on certain linear combinations of words.

Example

Find a function f on words such that $f(aB) + f(ba) - f(AA) = 1$. It's easy; set $f(aB) = 1$ and $f(ba) = f(AA) = 0$. But now suppose we also require $f(aB) + f(bb) - f(BA) = 1$, and so on. It becomes harder the more equations we require.

Finding quasimorphisms

It is not true that there is always an extremal counting quasimorphism of length 2 for a given word. However, sometimes they do exist. Let us try to find them.

Rotation quasimorphisms

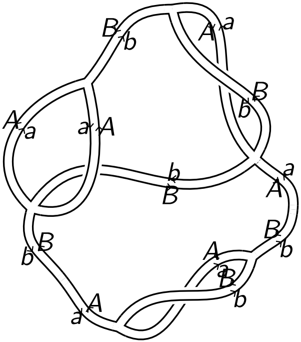
There is a special, very nice class of counting quasimorphisms of length 2 called *rotation quasimorphisms*. There are several ways of describing them.

Given a set of tripods of arm length 1, we want to find a counting quasi of length 2 that assigns 1 to all of them, and at most 1 to all other tripods (i.e. takes its defect on the given tripods).

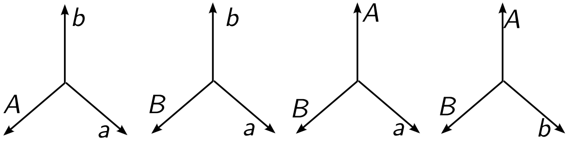
A *rotation quasimorphism* solves this problem when the tripods are *compatibly cyclically ordered*.

Rotation quasimorphisms example

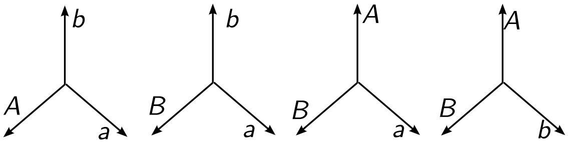
Here is the extremal surface for *aBAbaaBABAbb* again.



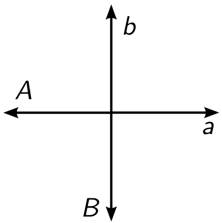
Here are some tripods which occur:



Rotation quasimorphism example

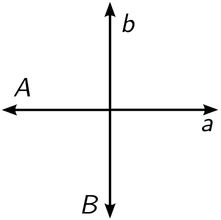


Notice that all of the tripods are *compatible* with the *cyclic order* on the letters a, A, b, B below:



Cyclic orders

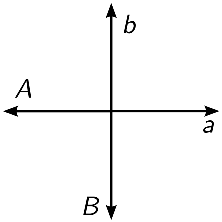
A *cyclic order* on a set S is an arrangement of the the elements of S around a circle, or equivalently, a map $o : S \times S \times S \to \{-1, 0, 1\}$. The value of $o(s_1, s_2, s_3)$ records whether s_1, s_2, s_3 goes around the circle positively or negatively (it's 0 if the triple is degenerate; for example $s_1 = s_2$, etc).



For example, for the cyclic order pictured, we have $o(a, b, A) = 1$ and $o(b, a, B) = -1$.

Rotation quasimorphisms from cyclic orders

Here is the cyclic order S again:

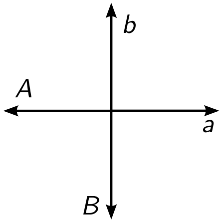


For the ordered pair (x, y) with $x, y \in S$, we define the *signed distance* $d_S(x, y)$ from x to y to be

$d_S(x, y) =$ distance x to y clockwise$-$distance x to y counterclockwise

E.g. $d_S(a, b) = 3 - 1 = 2$ and $d_S(a, A) = 0$.

Rotation quasimorphisms from cyclic orders



Then we define rot_S to be the quasimorphism

$$\text{rot}_S = \sum_{xy} d_S(x^{-1}, y) C_{xy}$$

where the sum is taken over all reduced words xy of length 2.

Rotation quasimorphisms from cyclic orders

$$\text{rot}_S = \sum_{xy} d_S(x^{-1}, y) C_{xy}$$

Now consider applying rot_S to a tripod which is oriented compatibly with the cyclic order S. The value will be

$$(Y+Z-X)+(Z+X-Y)+(X+Y-Z) = 2(X+Y+Z)-(X+Y+Z)$$

I.e. $X + Y + Z =$ constant.