MA1C ANALYTIC RECITATION 6/7/12 AND FINAL REVIEW

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1. Info

The exam has a time limit of **4.5 hours**. It is due **Wednesday, June 13 at noon**. It must be placed in the **final exam return slots**. Not the homework boxes. Not my mailbox. Not in the pile of old exams underneath the homework boxes.

It's open book, class notes, TA notes, your notes, posted solutions, but nothing you find online or whatever. You can use a computer for:

- Arithmetic
- Plotting stuff to help visualize
- Single-variable integrals

The last one means that if you have a 3-dimensional integral, we're making you do three one-variable integrals separately. We've got to make you do something!

You can find TA notes at:

- `http://www.its.caltech.edu/~awalker/12Ma1c/12Ma1c.html`
- `http://www.its.caltech.edu/~bhwang/ma1c/`

2. What you should know about

(1) Line integrals with respect to arc length (and the special case of line integrals)
(2) Gradient fields and their relationship to line integrals
(3) Green's theorem, both regular and for multiply connected regions
(4) Change of variables
(5) Surface integrals (and the special case of flux integrals)
(6) Curl, divergence, and Stokes' theorem

3. Definitions

3.1. **Change of variables.** Let $U \subseteq \mathbb{R}^n$, with coordinates $u_1, \dots, u_n$ and $V \subseteq \mathbb{R}^n$, with coordinates $v_1, \dots, v_n$, and let $\phi : U \to V$ be a diffeomorphism on U (up to a set of content zero). Let $f : V \to \mathbb{R}$ be an integrable function on V. Then

$$\int_V f dv_1 \cdots dv_n = \int_U (f \circ \phi)|D\phi| du_1 \cdots du_n$$

This is a theorem, because we already know how to integrate in both U and V.

3.2. **Line integrals with respect to arc length.** Let $c : [a, b] \to \mathbb{R}^n$ parameterize the curve C. Let $f : C \to \mathbb{R}$ be an integrable function defined on the curve C (perhaps defined on all of $\mathbb{R}^n$, for example). Then

$$\int_C f ds = \int_a^b f(c(t))\|c'(t)\| dt$$

This is a definition, because a priori we don't know how to integrate along the curve C.

In a special case, we define f on C by $f(c(t)) = F(c(t)) \cdot \left(\frac{1}{\|c'(t)\|} c'(t)\right)$, where F is some vector field. In this case,

$$\int_C f ds = \int_a^b f(c(t))\|c'(t)\| dt = \int_a^b F(c(t)) \cdot c'(t) dt$$

Which we denote by $\int_C F \cdot ds$.

3.3. **Surface integrals.** Let $\phi : \mathbb{R}^2 \to \mathbb{R}^3$ be a diffeomorphism to its image when restricted to a set $U \subseteq \mathbb{R}^2$, so $\phi : U \to S$, where S is some surface in $\mathbb{R}^3$. Let $f : S \to \mathbb{R}$ be some function defined on S (maybe defined on all of $\mathbb{R}^3$, for example). We denote the coordinates in the domain $\mathbb{R}^2$ by u and v. Then

$$\iint_S f dS = \iint_U (f \circ \phi) \left\| \frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v} \right\| dudv$$

This is a definition of a surface integral, because a priori we don't know how to integrate over S.

In a special case, we define f on S by $f(\phi(u,v)) = F(\phi(u,v)) \cdot \left(\frac{1}{\left\| \frac{\partial \phi}{\partial u}(u,v) \times \frac{\partial \phi}{\partial v}(u,v) \right\|} \left(\frac{\partial \phi}{\partial u}(u,v) \times \frac{\partial \phi}{\partial v}(u,v) \right) \right)$, otherwise known as $F \cdot \mathbf{n}$, where F is some vector field in $\mathbb{R}^3$. In this case, we have

$$\iint_S f dS = \iint_U (F \circ \phi) \cdot \left(\frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v} \right) dudv,$$

which we denote by $\iint_S F \cdot \mathbf{n} dS$

3.4. **Green's theorem.** Let $F = (P, Q)$ be a vector field in $\mathbb{R}^2$, and let R be some region in $\mathbb{R}^2$. Then

$$\int_{\partial R} F \cdot ds = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

Where ∂R is a *collection* of disjoint simple closed curves, given *the induced orientations*(!). This means the curves on the "inside" go clockwise and the curve on the "outside" goes counterclockwise. Another way to say this is to say that R has boundary C_1 on the outside, and $C_2, \ldots C_k$ on the inside, we parameterize all the curves counterclockwise, and we have

$$\int_{C_1} F \cdot ds - \sum_{i=2}^{k} \int_{C_k} F \cdot ds = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

3.5. **Stokes' theorem.** Let S be a surface in $\mathbb{R}^3$ bounded by the curve C. The curve C is given the orientation induced as $C = \partial S$ (as you walk around the curve, the top of the surface should be on your left). Then if F is a vector field in $\mathbb{R}^3$, we have:

$$\int_C F \cdot ds = \iint_S \text{curl}(F) \cdot \mathbf{n} dS$$

3.6. **Pappus' theorem.** Let S be a surface of revolution, and C its generating curve. Let s be the arc length of C and $\rho(s)$ the distance to the rotation axis from the point of C given by s, then

$$\text{Area}(S) = 2\pi \int_0^l \rho(s) ds$$

Where l is the length of C (see the book and homework 8 solutions). So for example, if we have a cylinder, that's generated by a straight vertical line, $\rho(s)$ is constant at r (the radius), and $l = h$, the height, so we get that the area is $2\pi rh$, which is correct.

Another way to state this is to say that the area is $2\pi Lh$, where L is the length of the curve and h is the distance from the centroid of the curve to the axis of rotation.

Pappus' theorem can be handy when it's tricky to parameterize the surface.

3.7. **Parameterizing surfaces and lines.** There's no algorithm here – the best way to do this is to picture it for yourself and then figure out how to get the formulas.

3.8. **Curl and div.** You know the definitions. Note that div, grad, and curl are related by the facts that

$$\text{div}(\text{curl}(F)) = 0$$

For all $\mathcal{C}^2$ vector fields F. Also,

$$\text{curl}(\text{grad}(f)) = 0$$

For all $\mathcal{C}^2$ functions f. Note this means that if F is the curl of another vector field, then its divergence is zero, and you get a similar fact from the second equation.

4. Example

Compute the line integral of $F(x, y, z) = (x^2, z, y)$ around the curve C of intersection of $x^2 + y^2 + z^2 = 1$ and $x + z = 0$, where C is counterclockwise when viewed from above the origin.

This is the intersection of a sphere and a plane. Let's parameterize it. Note that the plane goes through the origin, and we get it by rotating the xy plane about the y-axis. Rotation about the y-axis is the matrix

$$\begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

So what's θ? We want the image of the x axis to be inside our plane, so we want $\cos(\theta) + \sin(\theta) = 0$. Thus, $\theta = -\pi/4$ works. Therefore, define

$$A(x, y, z) = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

This transformation is just a rigid rotation of $\mathbb{R}^3$, so the circle we want is the image of the circle of radius 1 in the xy plane, i.e. $c(t) = (\cos(t), \sin(t), 0)$. This means our circle is $C(t) = A(c(t)) = \left(\cos(t)/\sqrt{2}, \sin(t), -\cos(t)/\sqrt{2}\right)$. Note this is the wrong orientation, so we'll have to multiply by -1. Now let's do the integral.

$$\begin{aligned} \int_C F \cdot ds &= \int_0^{2\pi} (\cos(t)^2/2, -\cos(t)/\sqrt{2}, \sin(t)) \cdot (-\sin(t)/\sqrt{2}, \cos(t), \sin(t)/\sqrt{2}) dt \\ &= \int_0^{2\pi} -\cos(t)^2 \sin(t)/2^{3/2} - \cos(t)^2/\sqrt{2} + \sin(t)^2/\sqrt{2}\ dt \\ &= 2^{-3/2}\left(\cos(t)^3/3\big|_0^{2\pi} - (t/2 + \sin(2t)/4\big|_0^{2\pi} + (t/2 - \sin(2t)/4\big|_0^{2\pi}\right. \\ &= 0 \end{aligned}$$

Or, we could note that $\mathrm{curl}(F) = 0$, and C bounds a disk S in $\mathbb{R}^3$ which is just the plane contained in the sphere. Therefore, we use Stokes' theorem, which says that $\int_S \mathrm{curl}(F) \cdot \mathbf{n} dS = \int_C F \cdot ds$. Since the first integral is zero, so is the second.

5. Example

Let V be the top half of the solid unit ball in $\mathbb{R}^3$. Compute

$$\iiint_V x^2 + y^2 + z^2\ dxdxdz$$

Clearly, we should use spherical coordinates. In this case, we note that V is the image of $[0, 1] \times [0, 2\pi] \times [0, \pi/2]$ in the (ρ, θ, ϕ) coordinates, so

$$\iiint_V x^2 + y^2 + z^2\ dxdxdz = \int_0^1 \int_0^{2\pi} \int_0^{\pi/2} \rho\rho^2 \sin\phi d\phi d\theta d\rho$$

Which is easy to compute as $\left(\frac{\rho^4}{4}\right|_0^1 \times (-\cos(\theta)|_0^{\pi/2} \times 2\pi = \pi/2$

6. Example

What's the area of an ellipse with equation $x^2 + xy + y^2 = 1$? We're going to use Green's theorem to find the area. First, we need to parameterize it. This is slightly tricky, but here's a nice way to do it. We want to make a linear substitution of x and y so the equation becomes of the form $ax^2 + by^2$.

Note that the equation is $\mathbf{x}^t A \mathbf{x} = 1$, where $A = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$. This should bring you back to your days of working with Hessian matrices. We want to find a new matrix P so that $P^t AP$ is diagonal. It's key that we have that transpose in there, so we want an orthogonal matrix (whose transpose is its inverse). By finding eigenvectors and eigenvalues, you find that

$$P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \quad \text{and} \quad P^{-1} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

and

$$P^t A P = \begin{bmatrix} 3/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

Thus, define $\phi : \mathbb{R}^2 \to \mathbb{R}^2$, where we use coordinates u and v in the first and x and y in the second, by $\phi(u, v) = P \begin{bmatrix} u \\ v \end{bmatrix}$. Note, then that the (u, v) pairs (write $\mathbf{u} = (u, v)$) which map to the ellipse satisfy:

$$1 = (P\mathbf{u})^t A(P\mathbf{u}) = \mathbf{u}^t \begin{bmatrix} 3/2 & 0 \\ 0 & 1/2 \end{bmatrix} \mathbf{u}$$

I.e. $3u^2/2 + v^2/2 = 1$. This is easy to parameterize as $c(t) = (\sqrt{2/3}\cos(t), \sqrt{2}\sin(t))$. Then a circle parameterizing the ellipse is

$$C(t) = P(c(t))^t = (\cos(t)/\sqrt{3} - \sin(t), \cos(t)/\sqrt{3} + \sin(t))$$

To use Green's theorem, we set $F = (P, Q) = (0, x)$, so $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$. Then

$$\begin{aligned} \text{Area} &= \int_0^{2\pi} (0, \cos(t)/\sqrt{3} - \sin(t)) \cdot (-\sin(t)/\sqrt{3} - \cos(t), -\sin(t)/\sqrt{3} + \cos(t))dt \\ &= \int_0^{2\pi} \cos(t)^2/\sqrt{3} - (4/3)\cos(t)\sin(t) + \sin(t)^2/\sqrt{3}dt \\ &= \frac{2\pi}{\sqrt{3}} \end{aligned}$$

We can get this another way. Let's do a change of coordinates, with P as the change of coordinates map. Note that DP (the derivative matrix of P) is equal to P, so the area distortion is *constant* at $\det(P)$, and $\det(P) = 1$, so the area should be the same! Let's do the area calculation in the $u - v$ coordinates, so

$$\begin{aligned} \text{Area} &= \int_0^{2\pi} (0, \sqrt{2/3}\cos(t)) \cdot (-\sqrt{2/3}\sin(t), \sqrt{2}\cos(t))dt \\ &= \int_0^{2\pi} 2\cos(t)^2/\sqrt{3} \\ &= \frac{2\pi}{\sqrt{3}} \end{aligned}$$

7. Example

Compute the flux of the vector field $F(x, y, z) = (x, 1 - y^2, z)$ through the set $x^2 + y^2 = 1$, $|z| \leq 1$. I.e. a cylinder of radius 1 and height 2 centered at the origin.

We can parameterize the cylinder using cylindrical coordinates as $f(\theta, z) = (\cos\theta, \sin\theta, z)$, which has derivative vectors: $\frac{\partial f}{\partial \theta} = (-\sin(\theta), \cos(\theta), 0)$ and $\frac{\partial f}{\partial z} = (0, 0, 1)$. Their cross product is $\frac{\partial f}{\partial \theta} \times \frac{\partial f}{\partial z} = (\cos(\theta), \sin(\theta), 0)$. Therefore, the surface integral is:

$$\begin{aligned} \iint_S F \cdot \mathbf{n} dS &= \int_{-1}^1 \int_0^{2\pi} (\cos\theta, 1 - \sin^2\theta, z) \cdot (\cos\theta, \sin\theta, 0) d\theta, dz \\ &= \int_{-1}^1 \int_0^{2\pi} \cos^2\theta + \sin\theta - \sin^3\theta d\theta dz \\ &= 2\int_0^{2\pi} \cos^2\theta + \sin\theta - \sin^3\theta d\theta \\ &= 2\pi \end{aligned}$$