MATH 1C RECITATION 4/5/12

ALDEN WALKER

1. Info

Me: I am Alden Walker. My email is awalker@caltech.edu, my office is Sloan 156, and my office hours are Sundays 8-9pm. There are 7 TAs with as many office hours, so check the course website to see if there is someone with the hours you would like. Feel free to email me to ask questions or to set up another time to meet.

Website: My website is `http://www.its.caltech.edu/~awalker`. I plan to hand out notes every week, and these notes will be available on my website, in addition to other material that I think might be useful.

Feedback: You can give me anonymous feedback on my website. Note that tqfr reports from last term are delayed and not very specific. If I am really bad at something and you don't tell me, I will continue to be oblivious!

Homework: is due Mondays at 10am. You get 1 free extension of a week **if you tell me in advance**. For extreme circumstances in your life, you need a note from the dean.

Grading: Is 40% homework, 30% midterm, and 30% final.

Citation: **Cite any theorems you use** and **check the hypotheses**. You will lose points if you do not do this!

Online Tomfoolery: There is none of this. Don't ask Dr. Math or Math Overflow.

2. Interesting Problem

Consider two numbers. Let T_n be the number of full rooted binary trees with $n+1$ leaves. Let W_n be the number of walks in the integer lattice in $\mathbb{R}^2$ from $(0,0)$ to (n,n) which do not go below the line $y = x$. Why is $T_n = W_n$?

Here is how you get a walk from a tree. I think you will see how to go backwards, although it is a little more tricky. Label the edges of a tree in the following way: on every edge going down and to the left, put a "U", and on every edge going down and to the right, put a "R". Now, do a depth-first traverse of the tree, writing down the labels of the edges as you pass them. You do not write down any edges after the first time you traverse them (or, say, only write down a label if you are going down the edge), and your depth-first traverse always goes to the left first. Notice that at the end of this you have a string like "UURURR." Translate this into a walk in the integer lattice by going up for every "U" and to the right for every "R." You go left first, so you always have more U's than R's, which means your walk does not go below the line $y = x$.

To go backward, it is possible to reconstruct the tree from the traverse labels, but it requires a little thought.

The numbers $T_n = W_n$ are called the Catalan numbers, and they count many interesting things. There is a nice formula which says that $T_n = W_n = \frac{1}{n+1}\binom{2n}{n}$.

3. Open Sets

A bunch of questions on your homework have to do with open sets in $\mathbb{R}^n$. I think you'll find these questions easier than the continuity questions, at least intuitively, but it is important, mostly for problem 2, to be careful when proving things are open. Here are some definitions—there are tons of various words thrown around in this subject (which is basically elementary point-set topology in $\mathbb{R}^n$, by the way).

- A set A in $\mathbb{R}^n$ is **open** if for every point $x \in A$ there exists a ball about $\mathbf{x}$ entirely contained in A.
- A set is **closed** iff its complement (in $\mathbb{R}^n$) is open.
- The above definition of closed is equivalent to: a set A is closed if it contains all of its limit points, where a limit point is the limit of a sequence contained in A.
- The **interior** of a set A is the largest open set contained in A.

Let's denote the ball of radius r about $\mathbf{x}$ by $B_r(\mathbf{x})$. That is, $B_r(\mathbf{x}) = \{\mathbf{a} \in \mathbb{R}^n \mid \|\mathbf{a} - \mathbf{x}| < r\}$.

3.1. **How To Prove Something Is Open.** I know you have gone through Math 1a and 1b, so you have a pretty good idea of how to rigorously prove things, but I think it's worth going over: basically every proof that a set A is open goes the same way: given $\mathbf{x} \in A$, you somehow produce an r such that you can show that $B_r(\mathbf{x}) \subseteq A$. Your method works for every $\mathbf{x}$ in A, perhaps by taking cases on what $\mathbf{x}$ is, so you have shown that every point has a ball about it, and thus A is open.

When you are asked to show that something is open with a "geometric argument," that means that you can appeal to common sense when saying things like "for every $\mathbf{x} = (x_1, x_2)$ with $x_1 \neq x_2$, there is some r such that $B_r(\mathbf{x})$ is disjoint from the line $x_1 = x_2$."

3.2. **Example 1.** The open interval $(0,1)$ is open, because given a point $x \in (0,1)$, we note that $B_{\min(x/2,(1-x)/2)}(x) \subset (0,1)$.

3.3. **Example 2.** The open unit ball $S = \{(x,y) \,|\, x^2 + y^2 < 1\}$ is open. Given some point $\mathbf{v} = (x,y)$, let $r = 1 - x^2 - y^2 = 1 - \|\mathbf{v}\|$. Then $B_{r/2}(\mathbf{v}) \subseteq S$: suppose we have $\mathbf{w}$ such that $\|\mathbf{v} - \mathbf{w}\| \leq r/2$. Then note

$$\begin{aligned}
\|\mathbf{w}\| &= \|\mathbf{w} - \mathbf{v} + \mathbf{v}\| \\
&\leq \|\mathbf{w} - \mathbf{v}\| + \|\mathbf{v}\| \\
&\leq r/2 + \|\mathbf{v}\| \\
&= \frac{1}{2}(1 - \|\mathbf{v}\|) + \|\mathbf{v}\| \\
&= \frac{1}{2}(1 + \|\mathbf{v}\|) \\
&\leq 1
\end{aligned}$$

3.4. **Example 3.** This is the general case of the previous example, and as above, it requires the triangle inequality: $B_r(\mathbf{x})$ is open, because if $\mathbf{y} \in B_r(\mathbf{x})$, then let $s = \|\mathbf{y} - \mathbf{x}\|$. Consider a point $\mathbf{z} \in B_{r-s}(\mathbf{y})$. We know

$$\begin{aligned}
\|\mathbf{z} - \mathbf{x}\| &= \|\mathbf{z} - \mathbf{y} + \mathbf{y} - \mathbf{x}\| \\
&\leq \|\mathbf{z} - \mathbf{y}\| + \|\mathbf{y} - \mathbf{x}\| \\
&= r - s + s \\
&= r
\end{aligned}$$

and thus $B_{r-s}(\mathbf{y}) \subseteq B_r(\mathbf{x})$. We can do this for any point $\mathbf{y} \in B_r(\mathbf{x})$, so we see that $B_r(\mathbf{x})$ is open. Sometimes it is harder than you would think to prove "trivial" things.

3.5. **(non) Example 4.** You have a homework problem similar to this. The rationals are not open. In fact, the interior of the rationals is empty. To see this, take any rational x and any radius r. Then we know that $B_r(\mathbf{x})$ contains an irrational number, so it is not contained in $\mathbb{Q}$. Thus, the rationals contain no open sets, so the interior is empty.

3.6. **Example 5.** A set can be open, closed, both, or neither. Don't be fooled! For example, the set $\mathbb{R}^n$ is open and closed. The interval $[0,1)$ in $\mathbb{R}$ is neither open nor closed.

3.7. **Example 6.** The complement of the image of the parametric function $f(t) = (e^{-t}\cos t, e^{-t}\sin t)$ for $t > 0$ is not open. Consider the point $(0,0)$ and choose any $r > 0$. Because f gets arbitrarily close to zero, $B_r((0,0))$ cannot be contained in the complement of the image. Specifically, given $r > 0$, let $t = -\log(r/2)$. Then

$$\begin{aligned}
\|f(t)\| &= \sqrt{e^{2\log(r/2)}\cos^2(-\log(r/2)) + e^{2\log(r/2)}\sin^2(-\log(r/2))} \\
&= \sqrt{e^{2\log(r/2)}} \\
&= r/2
\end{aligned}$$

Thus there are points on the graph inside the ball, so the ball isn't contained in the complement. This is true for all $r > 0$, so the complement isn't open.

4. Limits

I'm not sure if you have explicitly seen limits of sequences in higher dimensions. If $\mathbf{x}_i \in \mathbb{R}$ is a sequence, then we say that $\lim_{i\to\infty} \mathbf{x}_i = L$ iff for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $i > N$ we have $\|\mathbf{x}_i - L\| < \epsilon$. This is really the same as one dimensional limits, except now there's a norm instead of an absolute value.

4.1. **Example.** Let $\mathbf{x}_i = (1/i, 1/i^2)$. Then $\lim_{i\to\infty} \mathbf{x}_i = (0,0)$. Given $\epsilon > 0$, choose $N = 2/\epsilon$. Then if $i > N$, we have

$$\begin{aligned} \|\mathbf{x}_i\| &= \sqrt{1/(4/\epsilon^2) + 1/(16/\epsilon^4)} \\ &= \sqrt{\frac{1}{4}\epsilon^2(1 + \frac{1}{4}\epsilon^2)} \\ &< \sqrt{\frac{1}{4}\epsilon^2(2)} \\ &= \epsilon/\sqrt{2} < \epsilon \end{aligned}$$

Limits and continuity are very connected. You don't seem to have homework explicitly about continuity, but I'm leaving in the next section because I think it's important. You can think about it this way: if a function f has a limit at a point, then f can be made continuous at that point by setting its value to its limit. Let's do a straightforward example of limits. A function f has a limit L at a, written $\lim_{x\to a} f(x) = L$ (where x here might be a vector), if for all $\epsilon > 0$ there exists $\delta > 0$ such that if $\|x - a\| < \delta$, then $\|f(x) - f(a)\| < \epsilon$.

You show that something has a limit by exhibiting a δ for any given ϵ. Typically the best way to show that something does not have a limit at a is to find two sequences x_i and y_i approaching a such that $f(x_i)$ and $f(y_i)$ have different limits. This works because of following useful theorem:

Useful Theorem: $\lim_{x\to a} f(x) = L$ if and only if for all sequences x_i with $\lim_{i\to\infty} x_i = a$ we have $\lim_{i\to\infty} f(x_i) = L$.

Don't try to prove that something has a limit this way! You can't, since there are tons (uncountably many) sequences approaching a. However, it works great for the converse.

4.2. **Example.** Show that $\lim_{x\to(0,0)} f(x,y) = 0$, where $f(x,y) = x^2 + y^2$. Well, given $\epsilon > 0$, we want to find a δ such that $\|(x,y)\| < \delta$ implies $x^2 + y^2 < \epsilon$. You need to kind of work backwards like with one-dimensional limits. Choose $\delta = \sqrt{\epsilon}$. Then if $\sqrt{x^2 + y^2} = \|(x,y)\| < \delta = \sqrt{\epsilon}$, we have $x^2 + y^2 < \epsilon$.

4.3. **Example.** Show that $x^2/(x^2 + y^2)$ does not have a limit at $(0,0)$: let $v_i = (1/i, 0)$ and let $w_i = (0, 1/i)$. Then note that $f(v_i) = (1/i^2)/(1/i^2) = 1$, which goes to 1, but $f(w_i) = 0$, which goes to 0. Since these limits do not agree, f cannot have a limit at $(0,0)$. Note that something like $x^3/(x^2 + y^2)$ does have a limit, however, because the numerator approaches 0 fast enough.

5. Continuity

Continuity in 1 dimension is fairly straightforward, both intuitively and in its formal definition. In higher dimensions, however, the intuition still applies, but there are many tricky ways in which a function can be discontinuous.

5.1. **Definition.**

- A function $f : \mathbb{R}^n \to \mathbb{R}^m$ (we might also consider functions whose domain is a subset of $\mathbb{R}^n$) is **continuous** at a point $\mathbf{a} \in \mathbb{R}^n$ if $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$. A function which is continuous at all points is just called continuous. Really this definition means: the function has a limit at a point, and the value of the function is that limit. The easiest way to show that something isn't continuous is to show it doesn't have a limit.

Notice that this definition, translated into a verison that is useful for proving things, is: for all $\epsilon > 0$ there exists $\delta > 0$ such that $\|\mathbf{x} - \mathbf{a}\| < \delta \Rightarrow \|f(\mathbf{x}) - f(\mathbf{a})\| < \epsilon$. Just like with one-dimensional continuity, it is often very tedious and/or annoyingly difficult to prove that a function is continuous straight from the definition.

5.2. **How to Prove Something is Continuous.** Use pages 248–250! They outline lots of facts that you can put together to get many nice facts. For instance, is $x^2y^2/(x+1)$ continuous when $x \neq -1$? Yes! Because Example 5 on page 249 says so! Here is a basic outline of what to do in general when you are asked where a function is continuous (note this question implies you need to prove that it is continuous where you say AND that it is not continuous elsewhere!).

- Figure out for yourself where the function is continuous
- Use pages 248–250 to prove the vast majority of points are continuous, leaving a handful of points for you to deal with by hand.

- Deal with the leftover points: for each one, either prove the function is continuous there (with a limiting argument involving δ and ϵ), or show it is not continuous there (usually by exhibiting a sequence $x_n \to z$ such that $\lim_{n\to\infty} f(x_n) \neq f(z)$).
  This is especially easy if the function isn't defined at one of the leftover points! In that case, you don't have to prove anything for that point.

5.3. **Example 1.** Define $f(x,y) = (1/y^2)\sin(x)$, wherever the right hand side is defined. Where is f continuous?

Ok, so sin and x are defined and continuous for all real numbers. Also, $1/y^2$ is defined (and continuous) for all $y \neq 0$. By Theorems 8.1 and 8.2, we therefore know that $(1/y^2)\cos(x)$ is defined (and continuous) exactly on the set of points such that $y \neq 0$.

Note that in this example, the "hard" part was really figuring out where the function is defined—the continuity in this case came as an easy consequence of the theorems.

5.4. **Example.** Determine the set of points for which

$$f(x,y) = \begin{cases} \frac{x}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}$$

is continuous.

To do this, we use Theorem 8.2 to note that $\sqrt{x^2+y^2}$ is continuous away from $(0,0)$, because square root is continuous away from 0, and x^2+y^2 is continuous by the example on page 249. Therefore, we only need to determine the continuity at $(0,0)$. The function is *not* continuous at $(0,0)$. To see this, consider the sequence $x_n = (1/n, 0)$. Then $f(x_n) = \frac{1/n}{\sqrt{(1/n)^2+0}} = (1/n)/(1/n) = 1$. Therefore $\lim_{n\to\infty} f(x_n) = 1 \neq f((0,0))$, but $\lim_{n\to\infty} x_n = (0,0)$. We therefore conclude that the function is not continuous at $(0,0)$.

Now think about this: is there any value c so that we could set $f(0,0) = c$ and have f be continuous at $(0,0)$? By the above argument, $c = 1$, but if you take the sequence $y_n = (0, 1/n)$ you will see that $c = 0$, so that's a contradiction and no such c is possible.

5.5. **Example.** Is $xy\sin(1/xy)$ continuous at $(0,0)$, assuming that we define its value at $(0,0)$ to be 0? The answer is yes. Suppose that ϵ is given. Then choose $\delta = \min(\epsilon, 1)$. Then if $\|(a,b)\| < \delta$, we know that $|a| < \delta \leq 1$ and $|b| < \delta \leq 1$, so $|ab| < \delta < \epsilon$. As $|\sin(1/ab)| \leq 1$, we see that $|ab\sin(1/ab)| < \epsilon$, and thus the function is continuous at $(0,0)$.

This example is fairly straightforward, but it illustrates all the necessary pieces to a proof that a function is continuous.

5.6. **Tricky Example.** Usually, showing that a function is not continuous somewhere boils down to taking two sequences which approach that point and showing that the function evaluated on those sequences approaches different values. On most of the examples that you have seen, the sequences are easy to find, like perhaps along the x-axis and along the y-axis. Sometimes you might have to take along the line $y = x$ or something. Occasionally, you might have to be even trickier. Consider the function:

$$f(x,y) = \begin{cases} 1 & \text{if } |y| > x^2 \text{ or } y = 0 \\ 0 & \text{otherwise} \end{cases}$$

I admit this is a little contrived, but note the following: if you approach the point $(0,0)$ along *any* straight line, the limit is always 1. However, it is obvious that if you approach along the line $y = x^2$, the limit of the function is 0. This is a cautionary example. Verifying that the function approaches its value along all straight lines is not enough for continuity!

Another function which has this interesting property is $f(x,y) = \frac{x^2y}{x^4+y^2}$.

5.7. **Example.** Another interesting example is the function $f(x,y) = \arcsin\left(\frac{y}{\sqrt{x^2+y^2}}\right)$. This function isn't defined at $(0,0)$, and it's impossible to define $f(0,0)$ in such a way that it is continuous, because if you define a sequence x_n which approaches the origin along the line with angle θ from the x-axis, then $\lim_{n\to\infty} f(x_n) = \theta$!