MA1C ANALYTIC RECITATION 4/26/12

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1. Stirling's Formula and Computation Time

There is a very nice formula that you should be aware of; Stirling's formula, which says:

$$n! \approx \sqrt{2\pi n}\left[\frac{n}{e}\right]^n$$

Where the $\approx$ means that the two sides have the same limit as n goes to ∞. In practice, the relative error is quite low for pretty much all values of n. One interesting application of this is to provide a simple formula for the lower bound on computation time to sort a list. Sorting a list is the same thing as figuring out what permutation of the list puts it in order. Suppose that we consider one comparison to be single operation—each comparison returns $<$, $>$, or $=$, depending on the inputs. If we look at the computation tree for any algorithm which sorts n items, it is at most 4 valent (each vertex has at most 3 children), and there are at least $n!$ leaves at the bottom (corresponding to the $n!$ different permutations needed to put the items in order). This means that the depth of the tree is at least $\log_3 n!$. Using Stirlings formula, we see that this is

$$\log_3 n! = \log_3\left(\sqrt{2\pi n}\left[\frac{n}{e}\right]^n\right) = n\log_3 n - n(1+\log_3 e) + \frac{1}{2}\log_3 n + C$$

Therefore, we can see that sorting a list requires at least $O(n\log n)$ operations, so we know that algorithms such as mergesort are optimal, at least as far at time is concerned.

2. Lagrange Multipliers

It is common that you would want to optimize a function given some constraint. A nice way to do this is to use the method of Lagrange Multipliers: if you have a function $f : \mathbb{R}^n \to \mathbb{R}$ and constraints $g_1, \cdots g_m : \mathbb{R}^n \to \mathbb{R}$ (i.e. $g(\mathbf{x}) = 0, \cdots, g(\mathbf{x}) = 0$), then where f has a relative extremum subject to the constraints we have:

$$\nabla f = \lambda_1 \nabla g_1 + \cdots + \lambda_m g_m$$

This may seem arbitrary, but there is good intuition here: suppose that you are at some point in the domain of f. To increase the function, you want to follow the gradient. If the gradient points outside of the region to which you are constrained (the feasible region), then you can imagine being held in the feasible region and having the gradient be a force vector pulling you in that direction. Then even though the gradient points outside the feasible region, you will be pulled in the component of the gradient inside the feasible region, and you will continue to move until the gradient points in a direction perpendicular to the feasible region, at which point you are at a local maximum.

This method does not work if the ∇g_i are not independent. The book has an example in three dimensions on p. 317, but a simple example is to consider finding extrema of $f(x,y) = x^2 + y^2$ subject to the constraint $g(x,y) = x^2 - (y-1)^3 = 0$. Obviously, there is a local (actually global) minimum at $(0,1)$, but $\nabla g(0,1) = 0$, so there clearly is no λ such that $\nabla f(0,1) = (0,2) = \lambda \cdot 0$.

2.1. **Example.** Maximize the function $f(x,y) = x^3 - y^2$ over the set $g(x,y) = x^2 + y^2 = 1$. Well, $\nabla g(x,y) = (2x, 2y)$ and $\nabla f(x,y) = (3x^2, -2y)$. We must solve for where $g(x,y) = 1$ and $\nabla f = \lambda \nabla g$. This means: $-2y = \lambda 2y$, so either $\lambda = -1$ or $y = 0$. Consider $\lambda = -1$: then $3x^2 = (-1)2x$, so $x = 0$, and thus $y = 1$. Next, what if $y = 0$: then we must have $x = 1$. Therefore, our extrema subject to the constraint are $(0,1)$ and $(1,0)$. Checking our function, we have $f(0,1) = -1$ and $f(1,0) = 1$, so f takes a maximum value of 1 at $(1,0)$ subject to the constraint.

3. Mutiple Integrals

Just as with one-dimensional integration, multiple integrals are defined using step functions, except now the the step functions are functions that are constant on a finite number of rectangles (see p.354). If, for every pair of step functions s and t such that $s(\mathbf{x}) < f(\mathbf{x}) < t(\mathbf{x})$ for all $\mathbf{x}$ in the domain of integration R, we have $\int_R s \leq I \leq \int_R t$, and there is exatly one I with this property, then $\int_R f := I$.

Obviously, you don't integrate anything this way in practice, but it is important to remember this is the definition. There are a few theorems which help you actually perform integration. For instance (Theorem 11.5) if you assume that f is integrable in a region $Q = [a,b] \times [c,d]$ and that $\int_a^b f(x,y)dx$ exists for all y (call it $A(y)$), then if $\int_c^d A(y)dy$ exists, it is equal to $\iint_Q f$, i.e.

$$\iint_Q f(x,y)dxdy = \int_c^d \int_a^b f(x,y)dxdy$$

You should read Section 11.12 about regions and when functions are integrable over them.

3.1. **Example.** Assuming that f is integrable and $R = [0,1] \times [0,1]$, find $\iint_R ye^x dxdy$.

Let's apply Theorem 11.5, since we are assuming that our function is integrable. We note that $\int_0^1 ye^x dx$ exists for all y. In fact we see $A(y) = \int_0^1 ye^x dx = y(e-1)$. This function is also integrable on the interval $[0,1]$ and we see that $\int_0^1 A(y)dy = (e-1)\int_0^1 ydy = \frac{1}{2}(e-1)$. By Theorem 11.5, then, we know that $\iint_T ye^x dxdy = \int_0^1 \int_0, 1ye^x dxdy = \frac{1}{2}(e-1)$.

3.2. **Example.** Find the volume of the bounded solid bounded below by the xy-plane, above by $f(x,y) = x^2 + y^2$, and on the sides by the surface $x = y^2$ and plane $x = 1$.

There is only one bounded surface with these properties; the shadow in the xy-plane is a region of type 1, and f is continuous everywhere in $\mathbb{R}^2$, so f is integrable over this this region, and the integral can be computed as a double integral. It is important to think about the right way to parameterize this region. Let's let y go between -1 and 1. Then x runs from y^2 to 1, so our double integral (and the volume of the solid) is

$$\begin{aligned}
\int_{-1}^1 \int_{y^2}^1 x^2 + y^2 dxdy &= \int_{-1}^1 \left(\frac{x^3}{3} + y^2 x \bigg|_{y^2}^1 \right) dy \\
&= \int_{-1}^1 \left(\frac{1}{3} + y^2 - \frac{y^6}{3} - y^4 \right) dy \\
&= \left(\frac{y}{3} + \frac{y^3}{3} - \frac{y^7}{21} - \frac{y^5}{5} \bigg|_{-1}^1 \right. \\
&= \frac{2}{3} + \frac{2}{3} - \frac{2}{21} - \frac{2}{5} \\
&= \frac{88}{105}
\end{aligned}$$

3.3. **Example.** Show that a function f defined on $R = [0,1] \times [0,1]$ which is 1 at a finite number of (say n) points x_i and 0 elsewhere is integrable and has integral zero.

We have to go back to the definition to prove that it is integrable (soon we will have a theorem which will tell us this—you might already have done it in class). First, f is bounded. Next, we must show that if s and t are step functions such that $s \leq f \leq t$ we must have $\iint_R s \leq 0 \leq \iint t$, and 0 is the only number with this property. First, we show that 0 works in that inequality. Let s and t be step functions as above. Then take any rectangle in the definition of s. There must be some point $\mathbf{x}$ in that rectangle such that $f(\mathbf{x}) = 0$, so $s \leq 0$ in that rectangle. This holds for all rectangles in the definition of s, so s is everywhere less than zero. By the comparison Theorem (11.3), this means that $\iint s \leq 0$. Next, $0 = f \leq t$ by assumption, so by the Theorem 11.3 again, we know that $0 \leq \iint t$.

This shows that for any step functions s and t as above we have the desired inequality. We must now show that 0 is the only number for which that works. Suppose that we are given $\epsilon > 0$. Surround every x_i with the rectangle $R \cap [x_{i_1} - \sqrt{\epsilon/8n}, x_{i_1} + \sqrt{\epsilon/8n}] \times [x_{i_2} - \sqrt{\epsilon/8n}, x_{i_2} + \sqrt{\epsilon/8n}]$, and define a step function t to be 1 on those rectangles and 0 elsewhere. Additionally define $s \equiv 0$. Then $\iint t \leq n(2\sqrt{\epsilon/8n})^2 = \epsilon/2$. Thus $s \leq f \leq t$ but $\iint t < \epsilon$, so the required inequality does not hold for any $\epsilon > 0$. This shows that only 0 works, so f is integrable with integral 0. Note that technically we should check that no negative number could work either, but setting $s \equiv 0$ immediately discards those.

3.4. **Example - The Popcorn Function.** An interesting example of a function is the popcorn function (called by a bunch of different names). You will encounter something like this on homework. The popcorn function is defined on $[0,1]$ as:

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ in lowest terms} \\ 0 & \text{otherwise} \end{cases}$$

It has some interesting properties. It is discontinuous on $\mathbb{Q} \cap [0,1]$ and continuous on the irrationals. First, it is obviously discontinuous on the rationals because there is a sequence of irrational numbers approaching any rational. Next, if we consider any irrational number a, let $\epsilon > 0$ be given. The set of all the rational numbers with $1/q > \epsilon$ is a finite set, so let δ be the distance from a to the closest one of these finite points. Then if $|a - x| < \delta$, we must have $|f(x)| < \epsilon$.

Here is a warning: there is a theorem is your book which says that if a function has a set of discontinuities which is content zero (see the next section), then the function is integrable. We are about to see that the popcorn function is integrable, but you can think about it and see that the set of discontinuities of the popcorn function is not content zero. That is, the converse of the theorem isn't true.

Anyway, let's try to integrate the function. First, we guess that the integral is zero. Obviously, any step function which is less than f must be zero everywhere because in every interval there are irrational numbers. Similarly, any step function which is larger than f must be positive everywhere and thus have integral larger than zero. It remains to show that for any $\epsilon > 0$ we can find a step function t such that $f \leq t$ and $\int t < \epsilon$. There are a finite number of points $\{x_i\}_{i=1}^N$ such that $f(x_i) > \epsilon/2$. Let d be the minimum pairwise distance between these points, and let $D = \min\left(\frac{\epsilon}{2N}, \frac{d}{2}\right)$. Define t to be 1 on the intervals $[x_i - D/2, x_i + D/2]$ and $\epsilon/2$ elsewhere. Then $f \leq t$, and we see $\int t \leq ND + \epsilon/2 \leq \epsilon$. This proves that f is integrable and that $\int f = 0$.

4. Measure and Content

You will definitely feel more comfortable on the last homework problem if you have a feeling for measure and content.

- A set A has *content zero* if for all $epilson > 0$ there exists a finite collection of rectangles whose union contains A and whose total area is less than ϵ.
- A set A has *measure zero* if for all $epilson > 0$ there exists a (possibly infinite) collection of rectangles whose union contains A and whose total area is less than ϵ.

4.1. **Example (Measure and Content Zero).** Any finite collection of points x_i has both measure and content zero, because we can choose our rectangles to be only those points (e.g. $[x_i, x_i]$) in one dimension, and the total area of these rectangles is zero. If we want to require nontrival rectangles, then $[x_i - \epsilon/n, x_i - \epsilon/n]$, where there are n points, works.

4.2. **A Nontrivial Example.** Remember the Cantor set? Take $[0, 1]$ and remove the middle third, then remove the middle thirds of the remaining two sections, and repeat. Do this infinitely many times and take what's left over. This is the Cantor set (it's actually one of a family of Cantor sets, but it's the most common example). Now take as a sequence of collections of intervals the intervals that are left over after i iterations of the removing. These intervals contain the entire Cantor set, and it's clear that every time we remove one third of the area, so the collection at iteration i has total area $(2/3)^i$. This goes to zero, so we satisfy the requirement for being content zero. This implies that the Cantor set is measure zero also.

One reason that this might be a little counterintuitive is that, as noted when I talked about it before, the Canter set is uncountable, as it is in bijection with infinite binary expansions.

4.3. **Example (Measure Zero, Content Not Zero).** We just saw that you can have an uncountable content zero set. You can also have countable sets which are not content zero. For example, the rationals. If we want to cover the rationals in $[0, 1]$ with a finite number of intervals, we must clearly cover all of $[0, 1]$ and thus we cannot get our total area less than 1, let alone less than any ϵ.

However, the rationals have measure zero because they are countable. For any countable set, put it in bijection with the natural numbers, and around points x_i, put an interval of radius $\frac{\epsilon}{2}\left(\frac{1}{2}\right)^i$. Obviously this covers all of the points, and we can see that the total area is $\frac{\epsilon}{2}\sum_{i=1}^{\infty}\left(\frac{1}{2}\right)^i = \frac{\epsilon}{2}2 = \epsilon$. We can do this for any ϵ, so the set $\mathbb{Q} \cap [0, 1]$ has measure zero.

4.4. **Example (Measure and Content Not Zero).** Any interval (or rectangle in higher dimensions) has measure and content greater than zero.

4.5. **Integrability of functions with a content zero set of discontinuities.** This is Theorem 11.7 in your book:

Theorem 4.1. *Let f be defined and bounded on a rectangle $Q = [a, b] \times [c, d]$. If the set of discontinuities of f has content zero then the double integral $\iint_Q f$ exists.*

Note that this means that a function which has a finite number of discontinuties is integrable, as we showed in Example 3.3. The proof for content zero (not just finite) is basically the same.