MA1C ANALYTIC RECITATION 5/17/12

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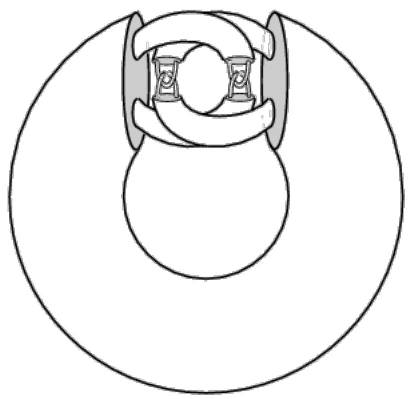
1. Alexander's horned sphere and simply connected sets

You have a homework problem this week to prove that a set is simply connected. There is a good definition of simply connected, but unfortunately, it's a little too complicated to get into. Therefore, for this course we will use the definition in Apostol (page 384): a set in $\mathbb{R}^2$ is simply connected if for every Jordan curve in the set, the inside component of the curve is also in the set.

So in order to do your problem, you need to show that the interior of any Jordan curve is contained in the set. A useful way to characterize the "interior" of a Jordan curve γ is that it's the component of $\mathbb{R}^2 \setminus \gamma$ which is bounded.

Note that all this uses the fact that a Jordan curve (a closed loop) divides the plane into two simply connected regions (topological balls). This is called Jordan's theorem, and its analogue (that a sphere in $\mathbb{R}^3$ divides $\mathbb{R}^3$ into two simply connected regions) is *not true*. Here is a picture of Alexander's horned sphere that I took from Wolfram MathWorld.:



This is a sphere, and the inside is a ball, but the outside is *not simply connected*!

2. Green's Theorem

Suppose you have some region $R \subseteq \mathbb{R}^2$ with a single (piecewise smooth) boundary curve C parameterized counterclockwise by c. Let $F(x,y) = (P(x,y), Q(x,y))$ be a vector field where P and Q are continuously differentiable scalar fields defined in an open set containing R. Then

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \int_C Pdx + Qdy$$

That's green's theorem. It can be helpful both ways, but it often turns out that the double integral is easier.

2.1. Example. Find the line integral of $\int_C F \cdot dc$, where $F(x,y) = (x^2+y, 3y^2-x)$, and C is the unit square R, traversed once counterclockwise.

To do this, we compute $\frac{\partial Q}{\partial x} = -1$ and $\frac{\partial P}{\partial y} = 1$. By Green's theorem, then

$$\int_C F \cdot dc = \iint_R -1 - 1dxdy = -2$$

We could even do a more complicated region–the beauty of Green's theorem is that you can wind up with a constant integrand. Say R were the unit disk and C the boundary. Then

$$\int_C F \cdot dc = \iint_R -2dxdy = -2\pi$$

Which we can check by setting $c(t) = (\cos t, \sin t)$, so $c'(t) = (-\sin t, \cos t)$, and

$$\begin{aligned}
\int_C F \cdot dc &= \int_0^{2\pi} (\cos^2 t + \sin t, 3\sin^2 t - \cos t) \cdot (-\sin t, \cos t)dt \\
&= \int_0^{2\pi} -\sin t \cos^2 t - \sin^2 t + 3\cos t \sin^2 t - \cos^2 t dt \\
&= \int_0^{2\pi} -\sin t \cos^2 t + 3\cos t \sin^2 t - 1dt \\
&= \left(\frac{-\cos^3 t}{3} + \sin^3 t - t \right|_0^{2\pi} \\
&= \frac{-1}{3} + 0 - 2\pi + \frac{1}{3} - 0 + 0 \\
&= -2\pi
\end{aligned}$$

So Green's theorem saved us a lot of work.

2.2. **Example – Area.** Green's theorem also gives a handy way of calculating the area enclosed by a curve. Choose any vector field such that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$. Then it's clear that the right hand side of Green's theorem will just give the area of the region. An example of a vector field which works is $F = \frac{1}{2}(-y, x)$ or $F = (-y, 0)$. For example, use my favorite example of the circle:

$$\int_0^{2\pi} (-\sin t, 0) \cdot (-\sin t, \cos t)dt = \int_0^{2\pi} \sin^2 t dt = \left(\frac{t}{2} - \frac{\sin 2t}{4} \right|_0^{2\pi} = \pi$$

3. Green's Theorem for "Multiply Connected" Regions

3.1. **Orientations.** This is kind of a tangent, but it applies to the general version of Green's theorem. I am specializing what I'm saying to $\mathbb{R}^n$, but this stuff generalizes. Notice that if you take a bunch of independent vectors and make them the columns of a matrix and take the determinant, you get something which is negative or positive. However, notice that if you wiggle the vectors in such a way that they are always linearly independent, the determinant never changes sign. To put it another way, the set of all invertible matricies (S, to use the notation on the midterm) has two separate components, those with positive determinant and those with negative determinant.

An orientation on a region of $\mathbb{R}^n$ is a choice of a basis of $\mathbb{R}^n$ at every point such that if you draw a line between any two points, the choice of basis at all points on that line is consistent, meaning that the determinant of the basis is either always positive or always negative. To see that the Mobius band is not invertible, pick a basis (for the tangent space) and walk once around the band. You will see that you cannot make a consistent choice that doesn't come around on itself without flipping sign.

Choosing a basis at every point is called "choosing a positive basis."

Now, if you have a region of $\mathbb{R}^n$ which has dimension n, then its boundary gets an induced orientation in the following way. Every point on the boundary has a canonical choice of a normal vector $\mathbf{n}$. A basis $\{\mathbf{x}_1, \cdots, \mathbf{x}_{n-1}\}$ for the boundary is positive if the basis $\{\mathbf{n}, \mathbf{x}_1, \cdots, \mathbf{x}_{n-1}\}$ is positive for the region. Since $\mathbb{R}^2$ has a canonical orientation (the standard one), boundaries of regions in $\mathbb{R}^2$ get induced orientations. This is why we always orient boundaries counterclockwise: if you're on an exterior boundary, the normal vector points out, and you can see by drawing a picture that the basis {normal vector, counterclockwise vector } gives a positive (standard) basis for $\mathbb{R}^2$. However, if you are on an inside boundary (like of an annulus), then the normal points in the "other" direction, and the boundary gets oriented clockwise.

This stuff will come up for Green's, Stokes', and the divergence theorems.

3.2. **General Green's Theorem.** Suppose that R is a region in $\mathbb{R}^2$ with a boundary a union of piecewise smooth Jordan curves and P and Q are continuously differentiable on an open set containing R. Then

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_{\partial R} Pdx + Qdy$$

Where the integral on the right hand side is understood to be a sum of integrals over the components of ∂R with their induced orientations. This can be specialized to the theorem in the book, where if C_1 is the "outside" boundary component and $\{C_k\}_{k=2}^N$ are the "inside" boundary components, then

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_{C_1} Pdx + Qdy - \sum_{k=2}^{N} \int_{C_k} Pdx + Qdy$$

Where all the C_k are traversed counterclockwise. The book needs to use the minus signs to correct for the fact that counterclockwise is the opposide of the induced orientation for the inside boundary components.

3.3. **Example.** What is the area of an annulus A with inner radius r and outer radius s. We already know the answer, but let's do with Green's theorem. Set $F = (-y, 0)$. Let C_r and C_s be the circles of radius r and s, traversed counterclockwise. Then

$$\begin{aligned}\iint_A 1dxdy &= r^2 \int_0^{2\pi} (-\sin t, 0)\cdot(-\sin t, \cos t)dt - s^2 \int_0^{2\pi} (-\sin t, 0)\cdot(-\sin t, \cos t)dt \\ &= (r^2 - s^2)\int_0^{2\pi} \sin^2 t dt \\ &= (r^2 - s^2)\pi\end{aligned}$$

Not a surprise, but certainly interesting. One way in which this shows up sometimes is that it is hard to integrate around one of the boundary components, but easy to do the others and to do the double integral. Then you can figure out what the remaining boundary integral is.

3.4. **Winding Number.** We have an intuitive notion about what it means for a curve to wind around a point, and it turns out there is a nice formula. If $c(t) = (X(t), Y(t))$ is a piecewise smooth closed curve on the interval $[a, b]$. Then the *winding number* of c around a point $p = (x_0, y_0)$ is:

$$W(c;p) = \frac{1}{2\pi}\int_a^b \frac{(X(t)-x_0)Y'(t) - (Y(t)-y_0)X'(t)}{(X(t)-x_0)^2 + (Y(t)-y_0)^2} dt$$

It is most instructive to look at this formula in the case that the circle is at the origin:

3.4.1. *Example.* Let $p = (0, 0)$ and let $c(t) = (\cos(2t), sin(2t))$ on the interval $[0, 2\pi]$. Then

$$\begin{aligned}W(c;p) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos(2t)2\cos(2t) - \sin(2t)(-2\cos(2t))}{\sin^2(2t) + \cos^2(2t)} dt \\ &= \frac{1}{2\pi} 2 \int_0^{2\pi} \frac{1}{1} dt \\ &= 2\end{aligned}$$

Not a big surprise. Basically, we are integrating the vector field $(-y, x)$ around the path, which intuitively should give the number of times we wind around counterclockwise.

3.5. **Example.** Suppose I have a vector field (P, Q) in $\mathbb{R}^2$ which so that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$ in some annulus. Then note that the line integral around either boundary of the annulus is the same!

4. Curl

Let ∇ denote the "vector" $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$. There are two things that are conveniently expressed using this. One is the curl of a vector field.

In $\mathbb{R}^3$, we define the curl of a vector field F to be

$$\text{curl}F = \nabla \times F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right), \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right), \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right)$$

Though this seems very odd, you will see that it is very useful. Maybe you have already seen why it is useful in physics. You can find an intuitive explanation in many places. The usual one is that if you stick a ball into a fluid moving with velocity described by the vector field, then the ball will rotate in the fluid. The curl of the vector field points along the axies of rotation, and the right hand rule gives the direction of the curl and its magnitude.

4.1. **Example.** You're going to see more interesting applications of curl soon, but for now you're just asked to compute it, so let's do that. What's the curl of $F = (y, 0, 0)$. It is

$$\text{curl}F = \nabla \times F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 0 & 0 \end{vmatrix} = (0, 0, 1)$$

This makes sense with the intuitive explanation from above if you draw a picture.

5. Divergence

I've always been a fan of divergence. The divergence of a vector field is defined for any dimension, but it is used mostly in $\mathbb{R}^3$, and is defined:

$$\text{div}F = \nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Notice that the divergence is a number, and the curl is a vector. Divergence sort of measures how much your vector field is expanding or contracting.

5.1. **Example.** The divergence of $F = (x^2, z^2, y^2)$ is $2x$.

5.2. **Example:** $\nabla \cdot (\nabla \times F)$**.** What is that? It's

$$\begin{aligned} \nabla \cdot (\nabla \times F) &= \left(\frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z}\right) + \left(\frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x}\right) + \left(\frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y}\right) \\ &= \left(\frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_3}{\partial y \partial x}\right) + \left(\frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_1}{\partial z \partial y}\right) + \left(\frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_2}{\partial x \partial z}\right) \\ &= 0 \end{aligned}$$

Where we need that F has continuous second partials for that to work.

In words, this says that the divergence of a curl is zero, or rather $\text{div}(\text{curl}(F)) = 0$.