MA1C ANALYTIC RECITATION 5/24/12

ALDEN WALKER

1. Euler Characteristic

I may have mentioned before that one central theme in lots of areas of math is that when you are confronted with a complicated object, you often want to obtain some simpler representation of it that still preserves some information. The simpler representation obtained from the complicated object is called an invariant. The main purpose is that it may be very difficult to tell apart two complicated objects, but if they are in fact the same object, they will produce the same invariant. If they produce two different invariants, then, you know that they are different.

One of the simplest invariants is Euler characteristic, which is just a number. Here is how you compute it. Suppose that your space X is made up of triangles that have been glued together along their edges. This is called a simplicial complex. Then the Euler characteristic $\chi(X)$ is the number of vertices, minus the number of edges, plus the number of faces. If you allow "tetrahedra" of any dimension (they are actually called simplices), then

$$\chi(X) = \sum_{k=0}^{\infty} (\text{number of simplices of dimension } k)(-1)^k$$

Let's see some examples. You can make a circle out of three edges and three vertices (just a triangle), so we compute $\chi(S^1) = 3 - 3 = 0$. We can make a sphere as the boundary of a tetrahedra, which has 4 vertices, 4 faces, and 6 edges, so $\chi(S^2) = 4 - 6 + 4 = 2$.

Is this really an invariant? Try to make up a triangulation of the sphere and see whether it always comes out to 2 (yes).

2. Change of Variables

Most of the homework this week is about doing double and triple integrals using changes of variables. All of this falls under one general theory, which it makes sense to talk about first. Afterwards, I'll do an example of each of the standard ones (polar, cylindrical, and spherical).

Here is the setup: you want to compute $\int \cdots \int_R f dV$ where $R \subseteq \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ is integrable, but the integral is difficult to compute, often because the limits of integration aren't natural in the coordinate system that you are using. Therefore, you pull back the integration using an diffeomorphism of $\mathbb{R}^n$. What this means is that you define a diffeomorphism (just meaning smooth, basically) $\varphi : \mathbb{R}^n \to \mathbb{R}^n$, or, since that isn't always possible for what you want, a diffeomorphism $\varphi : U \to R$, where $U \subseteq \mathbb{R}^n$. Then note that $f \circ \varphi : \mathbb{R}^n \to \mathbb{R}$ is integrable over U, and you want to compute

$$\int \cdots \int_R f dV = \int \cdots \int_U f \circ \varphi dV$$

But this is wrong! The problem is that φ (probably) distorts the volume of space, so you have to compensate for this. It turns out that $|\det D\varphi| = |J\varphi|$ is the right factor. Intuitively, this is because, as you may recall, the absolute value of the determinant is the volume of the parallelepiped spanned by the columns. Therefore the correct formula is:

$$\int \cdots \int_R f dV = \int \cdots \int_U f \circ \varphi |J\varphi| dV$$

A good way to think about this is that the integral is going to give a weighted sum of infinitismal rectangles in U: to sum them up, we take each rectangle, map it to R with φ, and weight it by the value of f. However, φ multiplies the volume of each rectangle by $|J\varphi|$, so we have to multiply by that factor.

2.1. **Example 1 - 1 dimension.** You probably don't think about it this way, but you are using this formula when you do a regular change of variables. It just so happens that in this case, the Jacobian is 1×1.

2.2. **Example 2 - Polar.** Define $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ by $\varphi(r, \theta) = (r\cos\theta, r\sin\theta)$. Then

$$D\varphi = \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix}$$

So $|J\varphi| = r(\cos^2\theta + \sin^2\theta) = r$. Therefore, let's integrate the function $f(x, y) = x^2 + y^2$ over the disk R of radius one. The disk is the image of $\varphi(U)$, where $U = [0, 1] \times [0, 2\pi]$ (see the note below). Therefore, we can write

$$\begin{aligned} \int_R f\,dA &= \int_0^1 \int_0^{2\pi} f \circ \varphi r\,d\theta\,dr \\ &= \int_0^1 \int_0^{2\pi} r^3\,d\theta\,dr \\ &= 2\pi \int_0^1 r^3\,dr \\ &= 2\pi \frac{1}{4} \\ &= \frac{\pi}{2} \end{aligned}$$

2.3. **Note on Boundaries.** Notice that φ actually is not a diffeomorphism on all of the domain, because it isn't injective (1-1) where $r = 0$, among other things. However, this doesn't matter. You almost never need to worry about the boundaries, because the boundaries have content (and measure) zero! What you need to do is the following: when you define a change of coordinates, make sure that it is 1-1 everywhere except some set S, and then check that the image $\varphi(S)$ has content zero in the image $\mathbb{R}^n$. On your homework, you just need to say something like "this is 1-1 except on the boundary, which doesn't matter since it is content zero."

2.4. **Example 3 - Cylindrical.** You can use the same idea as above to transform the first two coordinates in $\mathbb{R}^3$ while leaving the last one alone. The nice shapes in this coordinate system are cylinders, hence the name. The map φ in question is $\varphi(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$, which has Jacobian the same as polar, just with an extra row and column from the identity:

$$D\varphi = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Which has determinant r. Let's compute the integral of $x^2+y^2+z^2$ over the region R defined by $x^2+y^2 \leq 1$ and $|z| \leq 1$. We write:

$$\begin{aligned}
\int_R f\,dA &= \int_0^1 \int_0^{2\pi} \int_{-1}^1 f \circ \varphi r\,dz\,d\theta\,dr \\
&= \int_0^1 \int_0^{2\pi} \int_{-1}^1 (r^3 + rz^2)\,dz\,d\theta\,dr \\
&= \int_0^1 \int_0^{2\pi} \left(zr^3 + r\frac{z^3}{3} \right|_{-1}^1 d\theta\,dr \\
&= 2\pi \int_0^1 2r^3 + \frac{2r}{3}\,dr \\
&= 2\pi \left(\frac{r^4}{2} + \frac{r^2}{3} \right|_0^1 \\
&= 2\pi \left(\frac{1}{2} + \frac{1}{3} \right) \\
&= \frac{5\pi}{3}
\end{aligned}$$

2.5. **Example 4 - Spherical.** Next, let's do spherical. Here we define $\varphi(\rho, \theta, \phi) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)$. Here the Jacobian is:

$$D\varphi = \begin{bmatrix} \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \\ -\rho\sin\phi\sin\theta & \rho\sin\phi\cos\theta & 0 \\ \rho\cos\phi\cos\theta & \rho\cos\phi\sin\theta & -\rho\sin\phi \end{bmatrix}$$

Which has determinant $-\rho^2 \sin\phi$, so $|J\varphi| = \rho^2 \sin\phi$. Notice that the determinant was negative. You can see why this is the case if you use the right hand rule and see where the standard basis vectors in (ρ, θ, ϕ) space map to. Let's find the volume of half of a sphere S. Obviously, you could divide $4/3\pi r^3$ by 2, but let's check. The key thing here is the limits. We run ρ from 0 to r and ϕ from 0 to π, and then θ from 0 to π.

$$\begin{aligned}
\int_S dA &= \int_0^r \int_0^\pi \rho^2 \sin\phi\,d\phi\,d\rho \\
&= \pi \int_0^r \int_0^\pi \rho^2 \sin\phi\,d\phi\,d\rho \\
&= \pi \int_0^r \left(-\rho^2 \cos\phi \right|_0^\pi d\rho \\
&= \pi \int_0^r 2\rho^2 d\rho \\
&= 2\pi \left(\frac{\rho^3}{3} \right|_0^r \\
&= \frac{2}{3}\pi r^3
\end{aligned}$$

So yup that's right.

2.6. **Example 5 - Other Coordinates.** Sometimes, if you want to find the volume of some weird shape, the best thing to do it so figure out how to parameterize $\mathbb{R}^n$ so that your shape is the image of a nice set. For example, let's find the volume of the right angled intersection between two cylinders of radius r. This is an interesting shape: from the top, it looks like a square, and from the sides it looks like a circle. You can do this integral in standard coordinates, but let's illustrate how a change of coordinates can do it.

Set $\varphi(a, b, c) = (a\sqrt{r^2 - c^2}, b\sqrt{r^2 - c^2}, c)$. Notice that this is a diffeomorphism when $|c| < 1$. Then the intersection of two cylinders of radius r is the image under φ of $[-1, 1] \times [-1, 1] \times [-r, r]$. The Jacobian of

φ is:

$$D\varphi = \begin{bmatrix} \sqrt{r^2-c^2} & 0 & 0 \\ 0 & \sqrt{r^2-c^2} & 0 \\ \frac{-2ac}{\sqrt{r^2-c^2}} & \frac{-2bc}{\sqrt{r^2-c^2}} & 1 \end{bmatrix}$$

But those nasty terms don't make it into the determinant, which is just $J\varphi = r^2 - c^2$. Therefore the volume in question is:

$$\begin{aligned} \int_R dA &= \int_{-1}^{1}\int_{-1}^{1}\int_{-r}^{r} (r^2-c^2)\, dc\, da\, db \\ &= 4\int_{-r}^{r} (r^2-c^2)\, dc \\ &= 4\left(cr^2 - \frac{c^3}{3} \right|_{-r}^{r} \\ &= 4\left(r^3 - \frac{r^3}{3} - (-r)^3 + \frac{(-r)^3}{3} \right) \\ &= 4\left(2r^3 - \frac{2r^3}{3} \right) \\ &= \frac{16r^3}{3} \end{aligned}$$

Doing it in normal coordinates actually comes out to the same integral, but this seems nicer somehow. It think it's interesting that something which looks like a circle from two different directions has a volume which isn't a multiple of π.

Suppose now that instead of just getting the volume we wanted to integrate something over this region. An integrand like x^2+y^2 is easy to do these coordinates because it is just the polynomial $(a^2+b^2)(r^2-c^2)$. While certainly still possible in regular coordinates, you'd have intermediate terms involving $(r^2-c^2)^{\frac{3}{2}}$, and who wants to deal with that.

2.7. **Note About Cylindrical vs. Spherical.** It is often easier to use cylindrical coordinates when you are finding the volume of intersections and stuff. This is because while spherical coordinates are very nice for certain kinds of chunks of spheres, they don't work very well for caps of spheres. The example above in these notes gives you half of a sphere, but if you try to parameterize the piece of a sphere cut off by a plane that doesn't go through the center, you will find it quite difficult. Doing it in normal coordinates is also annoying because of the square roots that pop up. However, cylindrical coordinates usually take care of the square root by hiding it inside r.