MA1C ANALYTIC RECITATION 5/31/12

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1. Manifolds, etc

A manifold M is a topological space which looks locally like $\mathbb{R}^n$. Formally, this means that for every point $x \in M$, there exists an open set U containing x which is homeomorphic to an open subset of $\mathbb{R}^n$. Homeomorphic here means that that there is some $V \subseteq \mathbb{R}^n$ such that there exists $\varphi : U \to V$ such that φ is continuous, bijective, and has a continuous inverse. The n here is said to be the dimension of the manifold. The intuitive picture is much more clear: if you zoom way in to M, then it looks flat like $\mathbb{R}^n$. Examples of manifolds are spheres (of any dimension), a torus with any number of holes, a klein bottle, etc. The easiest examples to think about are surfaces, because you can often picture them in 3 dimensions. There are manifolds of any dimension, though: try to imagine $S^1 \times S^2$, which is a hollow sphere where the inside and outside are identified. Even though you can't really picture it without it folding in on itself, you can see that it is a 3-manifold.

You can also define a manifold with boundary, which is exactly what you think it is. The reason I mention these things is that you are dealing with manifolds all the time. When you parameterize a surface, you are in fact finding the function φ in the definition above! That is, you are proving that the surface is a manifold.

This digression logically continues later when talking about notation for surface integrals, so I'll pick it up then.

2. Surface Area

The right way to think about this whole situation is to continue thinking in terms of change of variables. Let's say we want to find the surface area of some chunk of surface S in $\mathbb{R}^3$. Now, suppose that we can find a function $\varphi : \mathbb{R}^2 \to S$ which takes some nice set U in $\mathbb{R}^2$ bijectively (and continuously) to S. Then we should just integrate over U to find the volume of S, except that would be wrong, for the same reason it was wrong in change of variables. The function φ distorts area when it maps U to S, so we need to figure out how it changes area at every point. With change of variables, we could use the determinant of the Jacobian matrix, but here that doesn't work because our image set S lives in $\mathbb{R}^2$ while our domain lives in $\mathbb{R}^2$ (so the derivative map $D\varphi$ is 3×2. So what do we do? We use the cross product. Recall that if you take vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, then $\|\mathbf{x} \times \mathbf{y}\|$ is the area of the parallelogram spanned by $\mathbf{x}$ and $\mathbf{y}$. Therefore, suppose that we take $U \subseteq \mathbb{R}^2$ to be (u, v)-space, then:

$$\text{Area}(S) = \iint_U (\text{Area of infinitismal rectangle spanned by } \frac{\partial \varphi}{\partial u} \text{ and } \frac{\partial \varphi}{\partial v})\, du\, dv = \iint_U \left\| \frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \right\| du\, dv$$

As an aside, an easy(ish) way to see that the length of the cross product gives the correct area is to notice that since the cross product is perpendicular to both $\mathbf{x}$ and $\mathbf{y}$, the area of the parallelogram is equal to the volume of the parallelopiped spanned by $\mathbf{x}$, $\mathbf{y}$, and $\frac{1}{\|\mathbf{x}\times\mathbf{y}\|}(\mathbf{x} \times \mathbf{y})$. That scaling factor makes the length of the last vector 1. The volume of this parallelopiped is the determinant of the matrix with rows $\mathbf{x}$, $\mathbf{y}$, and $\frac{1}{\|\mathbf{x}\times\mathbf{y}\|}(\mathbf{x}\times\mathbf{y})$. If you think about this, you'll notice that this determinant is exactly $\frac{1}{\|\mathbf{x}\times\mathbf{y}\|}\|\mathbf{x}\times\mathbf{y}\|^2 = \|\mathbf{x}\times\mathbf{y}\|$.

2.1. **Example.**

Parameterize the sphere by spherical coordinates with $\rho = 1$, i.e. $f(\theta, \phi) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$. Here I am using f to avoid two different phi's. Then notice $f([0, 2\pi] \times [0, \pi]) = S^2$. The overlap is a set of content zero in the domain and range, so our calculations will be correct. Thus, we calculate

$$\begin{aligned}\left\|\frac{\partial f}{\partial \theta} \times \frac{\partial f}{\partial \phi}\right\| &= \|(-\sin\phi\sin\theta, \sin\phi\cos\theta, 0) \times (\cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi)\| \\ &= \left\|(-\sin^2\phi\cos\theta, -\sin^2\phi\sin\theta, -\cos\phi\sin\phi)\right\| \\ &= |\sin\phi|\sqrt{\sin^2\phi\sin^2\theta + \sin^2\phi\cos^2\theta + \cos^2\phi} \\ &= |\sin\phi|\end{aligned}$$

And $\int_0^\pi \int_0^{2\pi} |\sin\phi|\, d\theta\, d\phi = 2\pi\left(-\cos\phi|_0^\pi\right. = 4\pi$, which we know is correct. Notice that while this one simplified down, we have the same problem that we did with line integrals with respect to arc length, which is that square roots can pop up and make our integrals nasty.

3. Surface Integrals

To integrate a function over a surface, then, we again follow the same ideas as in change of variables: with notation as before, to integrate f over S, we integrate f over U, weighted (at every point) by the factor that φ distorts the area of U. That is,

$$\iint_{\varphi(U)=S} f dS = \iint_U f(\varphi(u,v)) \left\|\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v}\right\| du\, dv$$

3.1. **Flux.** A special case of the function f is $f = F \cdot \mathbf{n}$, where F is a vector field in $\mathbb{R}^3$ and $\mathbf{n}$ is the normal vector to the surface. Like line integrals, these are actually easier that regular surface integrals, because

$$\mathbf{n} = \frac{\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v}}{\left\|\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v}\right\|}$$

So

$$\iint_{\varphi(U)} F \cdot \mathbf{n} dS = \iint_U F(\varphi(u,v)) \cdot \left(\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v}\right) dudv$$

Which doesn't have any square roots!

3.2. **Example.** Let's find the flux of the field $F(x,y,z) = (x,y,z)$ through the sphere of radius 1. We parameterize the sphere with function $g(\theta,\phi) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$ as above, but here our integrand is:

$$\begin{aligned}\iint_{S^2} F \cdot \mathbf{n} dS &= \int_0^{2\pi}\int_0^\pi F(g(\theta,\phi)) \cdot \left(\frac{\partial f}{\partial \theta} \times \frac{\partial f}{\partial \phi}\right) d\phi\, d\theta \\ &= \int_0^{2\pi}\int_0^\pi (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi) \cdot (-\sin^2\phi\cos\theta, -\sin^2\phi\sin\theta, -\cos\phi\sin\phi)\, d\phi\, d\theta \\ &= \int_0^{2\pi}\int_0^\pi -\sin^3\phi\sin^2\theta - \sin^3\phi\cos^2\theta - \cos^2\phi\sin\phi\, d\phi\, d\theta \\ &= \int_0^{2\pi}\int_0^\pi -\sin\phi\, d\phi\, d\theta \\ &= 2\pi\left(\cos\phi|_0^\pi\right. \\ &= -4\pi\end{aligned}$$

Why is this negative? Remember how the usual spherical coordinates are orientation reversing? Well, that's what's going on here. We have parameterized the sphere so that the unit normal vector points inward. That's perfectly fine; you just have to keep in mind which way the normal vector is pointing for your parameterization so that it coincides with what you want.

3.3. Other Notation for Surface Integrals, and the Digression Continued. First, we define:

$$\frac{\partial(X,Y)}{\partial(u,v)} := \begin{vmatrix} \frac{\partial X}{\partial u} & \frac{\partial Y}{\partial u} \\ \frac{\partial X}{\partial v} & \frac{\partial Y}{\partial v} \end{vmatrix}$$

And we define:

$$\iint_S P dy \wedge dz := \iint_T P(\varphi(u,v)) \frac{\partial(Y,Z)}{\partial(u,v)} dudv$$

Thus if you are given some integral in terms of the symbols $dx \wedge dy$, etc, you just need to break it up into summands and use the above pattern to tell you how to actually do the integral. I think this will be all you need in order to understand what problems 12.10.6 and 12.10.7, for instance, are asking you to prove.

3.3.1. *What's the Deal with the $\wedge$?* The wedge notation seems like it just introduces more complication into your lives. If you just want to do the homework, you use the key above to see what it is asking you to calculate, and you don't need to think about it. However, there is a reason why it is used. This will go a little fast, and don't worry if you get confused. The key is to focus on your intuition to see why things work and to resort to mindless calculation only when you need to actually calculate something.

Think about a surface M in $\mathbb{R}^3$. Attached to every point x in M there is a tangent space T_xM. Intuitively, this is just the space of vectors you get by taking a tangent plane to the surface and thinking of x as the origin. If M is parameterized by φ, then T_xM is spanned by (basically, it is *defined* as the span of) $\frac{\partial\varphi}{\partial u} := (\frac{\partial\varphi_1}{\partial u}, \frac{\partial\varphi_2}{\partial u}, \frac{\partial\varphi_3}{\partial u})$ and $\frac{\partial\varphi}{\partial v} := (\frac{\partial\varphi_1}{\partial v}, \frac{\partial\varphi_2}{\partial v}, \frac{\partial\varphi_3}{\partial v})$. Notice that I am basically assuming that M is a subset of $\mathbb{R}^n$ for some n, since I am differentiating and everything. There is a theorem (the Whitney embedding theorem) which says that this is ok, but technically M doesn't have to be a subset of $\mathbb{R}^n$ and it is possible to define this stuff in general. The intuition is a lot harder though.

So along with a parameterization, you get at every point a basis for the tangent space. Since writing down lots of partials is annoying, a nice notation to use for $\frac{\partial\varphi}{\partial u}$ is ∂_u. At every point, you can also consider the dual of the tangent space, i.e. the (vector space of) linear functions on T_xM. Let's call this T_x^*M. Now T_x^*M is spanned by the characterisic functions of the basis vectors. Notation for this is du, so for instance $du(\partial_u) = 1$, and $du(\partial_v) = 0$. A *1-form* on M is a (smooth, in a sense that I won't define) choice of a cotangent vector in every cotangent space T_x^*M.

A good way to think about a 1-form α is that for every point $x \in M$, you give α a tangent vector, and it gives you a number. For example, suppose our manifold is all of $\mathbb{R}^3$, and our 1-form is just defined to be dy everywhere. Then if we pick a point $x \in \mathbb{R}^3$ and a vector $v \in T_x\mathbb{R}^3 \cong \mathbb{R}^3$, $dy(v)$ is just the component of v int the y direction.

A k-form is a smooth choice over M of an element of $\bigwedge^k T_x^*M$ for all x. If you don't know what that means, don't worry. All you need to know is that the symbol $du \wedge dv$ means: a function on $T_xM \times T_xM$ which is antisymmetric and bilinear; defined by $(du \wedge dv)(a,b) = \frac{1}{2}(du(a)dv(b) - du(b)dv(a))$. A function f is bilinear if $f(a(\mathbf{x}+\mathbf{z}), b\mathbf{y}) = abf(\mathbf{x},\mathbf{y}) + abf(\mathbf{z},\mathbf{y})$ (the same equality holds for addition in the section coordinate). A function f is antisymmetric if $f(\mathbf{x},\mathbf{y}) = -f(\mathbf{y},\mathbf{x})$. Higher wedges $dx \wedge dy \wedge dz$, etc are defined similarly.

To check and see whether you understand everything, think about why it makes sense to multiply a real-valued function by a form.

Now, if you have an n-form α defined on an n-manifold M which is parameterized by φ defined on a set U, there is a natural way of defining what it means to integrate it over the manifold. Specifically $\int_M \alpha = \int_U \alpha(\varphi(u,v), \partial_u, \partial_v)\, du\, dv$. Here I am writing out for $n = 2$, and while α is really a function only on $T_xM \times T_xM$, I'm putting that first coordinate there to emphasize that $\alpha(\partial_u, \partial_v)$ depends on which point $\varphi(u,v)$ we are at.

This is all you need to decode what $\iint_S P dy \wedge dz$ means using the real definitions. Let all notation be as above, and then we plug in to the definition of integration above to get

$$\iint_S P dy \wedge dz \quad = \quad \iint_U P(\varphi(u,v))(dy \wedge dz)(\partial_u, \partial_v)\, du\, dv$$

So what is $(dy \wedge dz)(\partial_u, \partial_v)$? Let's expand it out and use the bilinearity and antisymmetry to figure it out:

$$\begin{aligned}
(dy \wedge dz)(\partial_u, \partial_v) &= (dy \wedge dz)\left(\frac{\partial \varphi_1}{\partial u}(1,0,0) + \frac{\partial \varphi_2}{\partial u}(0,1,0) + \frac{\partial \varphi_3}{\partial u}(0,0,1), \frac{\partial \varphi_1}{\partial v}(1,0,0) + \frac{\partial \varphi_2}{\partial v}(0,1,0) + \frac{\partial \varphi_3}{\partial v}(0,0,1)\right) \\
&= \frac{\partial \varphi_2}{\partial u}\frac{\partial \varphi_3}{\partial v}(dy \wedge dz)((0,1,0),(0,0,1)) + \frac{\partial \varphi_3}{\partial u}\frac{\partial \varphi_2}{\partial v}(dy \wedge dz)((0,0,1),(0,1,0)) \\
&= \frac{\partial \varphi_2}{\partial u}\frac{\partial \varphi_3}{\partial v} - \frac{\partial \varphi_3}{\partial u}\frac{\partial \varphi_2}{\partial v} \\
&= \frac{\partial(\varphi_2, \varphi_3)}{\partial(u,v)}
\end{aligned}$$

That last thing we can write in the usual way as $\frac{\partial(Y,Z)}{\partial(u,v)}$, so we see that the integral comes out to the formula given above.

3.4. **Example with $\wedge$.** Suppose that S is the surface of the chunk of cylinder $x^2 + y^2 = 1$ and $|z| < 1$, oriented so that the normal points outward. Compute $\iint_S xz^2 dy \wedge dz$.

Let's use cylindrical coordinates $\varphi(\theta, z) = (\cos\theta, \sin\theta, z)$. Then $\partial_\theta \times \partial_z = (-\sin\theta, \cos\theta, 0) \times (0,0,1) = (\cos\theta, \sin\theta, 0)$. Notice that this always points out (it is the same vector field as (x, y)), so we have oriented it correctly. Then we apply our formula, which says

$$\begin{aligned}
\iint_S xz^2 dy \wedge dz &= \int_0^{2\pi}\int_{-1}^1 \cos\theta z^2 \frac{\partial(Y,Z)}{\partial(\theta,z)}\, dz\, d\theta \\
&= \int_0^{2\pi}\int_{-1}^1 \cos\theta z^2 \left(\frac{\partial Y}{\partial \theta}\frac{\partial Z}{\partial z} - \frac{\partial Z}{\partial \theta}\frac{\partial Y}{\partial z}\right) dz\, d\theta \\
&= \int_0^{2\pi}\int_{-1}^1 \cos\theta z^2 \left(\cos\theta - 0\right) dz\, d\theta \\
&= \int_0^{2\pi}\int_{-1}^1 \cos^2\theta z^2\, dz\, d\theta \\
&= \int_0^{2\pi} \cos^2\theta \left(\left.\frac{z^3}{3}\right|_{-1}^1\right) d\theta \\
&= \frac{2}{3}\int_0^{2\pi} \cos^2\theta\, d\theta \\
&= \frac{2}{3}\left(\left.\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right|_0^{2\pi}\right. \\
&= \frac{2}{3}\pi
\end{aligned}$$

4. Stokes' Theorem

Stokes' theorem is on page 438. It is expressed there in a form with the wedges, so it is pretty incomprehensible. The usual form of it is that if S is a surface with boundary curve C, where S has an orientation (and thus a normal vector) and the orientation on C is induced from the one on S, and F is a vector field defined in a neighborhood of S, then

$$\iint_S (\nabla \times F) \cdot \mathbf{n} dS = \int_C F \cdot ds$$

I might mention that all the main theorems you have learned about integrals (Stokes', Green's, Divergence) are all special cases of a big famous theorem called Stokes' theorem, which reads

$$\int_M d\alpha = \int_{\partial M} \alpha$$

Where α is an $(n-1)$-form, and $d\alpha$ is called the exterior derivative, which is just a way of getting an n-form from an $(n-1)$-form. To specialize it to get all the various theorems, you just need to apply it to specific examples. It's very confusing. But we don't have to worry about that—just use the regular version of Stokes' theorem. As evidence that I am telling the truth, you might notice that all of the theorems have the form (integral of some crazy derivative object over a region) = (integral of something less complicated over the boundary), which is the form of Stokes' theorem.

4.1. **Example.** Let's find the line integral of $F(x, y, z) = (y^2, x^2, xz)$ around the circle of radius 1 in the xy-plane, oriented counterclockwise from above. The curl of F is $(0, z, 2y - 2x)$. By Stokes' theorem, the integral we want is equal to the integral of $\nabla \times F$ over any surface with C as a boundary. The obvious choice is a disk in the xy-plane, for which we can use polar coordinates, so

$$\begin{aligned}
\int_C F \cdot ds &= \int_0^1 \int_0^{2\pi} (\nabla \times F)(\varphi(r,\theta)) \cdot (\partial_r \times \partial_\theta)\ d\theta\, dr \\
&= \int_0^1 \int_0^{2\pi} (0, 0, 2r(\sin\theta - \cos\theta)) \cdot ((\cos\theta, \sin\theta, 0) \times (-r\sin\theta, r\cos\theta, 0))\ d\theta\, dr \\
&= \int_0^1 \int_0^{2\pi} (0, 0, 2r(\sin\theta - \cos\theta)) \cdot (0, 0, r)\, d\theta\, dr \\
&= \int_0^1 \int_0^{2\pi} 2r^2(\sin\theta - \cos\theta))\, d\theta\, dr \\
&= \int_0^1 \left(-2r^2(\sin\theta + \cos\theta)\right|_0^{2\pi}\ dr \\
&= \int_0^1 0\, dr \\
&= 0
\end{aligned}$$

4.2. **Example.** The flux of a curl of a vector field through any closed orientable (compact) surface is zero. Why? It has no boundary, so apply Stokes' theorem, the right hand side must be zero. I'm not sure if technically by the books definition a surface without boundary is allowed. In this case, you can still use the same argument, but to apply the theorem you can cut along any closed curve. What's left over is a surface with one or two components and two boundary components, which are C in both orientations. The line integral along two copies of C with opposite orientations is always zero, so the flux integral is zero.

4.3. **Example.** Let's evaluate the surface integral $\iint_S (\nabla \times F) \cdot n\, dS$, where S is the top hemisphere of the unit sphere, and $F = (x^2, xy, xz)$ (this is 12.13.1): By Stokes' theorem, the integral of the curl is the line integral of F around the boundary. Note that the orientation is counterclockwise looking from above, so we parameterize it by $c(t) = (\cos(t), \sin(t), 0)$ for $t \in [0, 2\pi]$. The line integral is then

$$\begin{aligned}
\int_0^{2\pi} F \cdot ds &= \int_0^{2\pi} (\cos^2(t), \cos(t)\sin(t), 0) \cdot (-\sin(t), \cos(t), 0) dt \\
&= \int_0^{2\pi} \cos^2(t)\sin(t) - \cos^2(t)\sin(t)\, dt \\
&= 0
\end{aligned}$$