MATH 2A FINAL REVIEW 12/4/11

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1. Info

The final is due Thursday 12/8 at noon. It is open book, open note, open recitation notes, etc. Not open google, mathoverflow, etc. Except for one problem, you can use a computer to draw phase portraits. You shouldn't need to do difficult integrals.

2. Things to know about

(1) Undetermined coefficients
(2) Oscillations
(3) Moving back and forth between linear systems and higher order differential equations
(4) Phase portraits:
  (a) Phase portraits of linear systems
  (b) Linearization of nonlinear systems
  (c) Deciding about the centers
(5) Ecological models
(6) Conservative and dissipative systems
(7) Lyapunov functions
(8) Boundary value problems
(9) Applications to PDEs
  (a) Fourier series
  (b) Solving PDEs

3. Review example - Undetermined coefficients

If the equation $L[y] = g$ has constant coefficients, and g is a quasi polynomial, then we can write the form of a particular solution. Here, quasi polynomial means that g is a linear combination of functions of the form:

$$t^m e^{\mu t}, \quad t^m e^{\alpha t}\cos(\omega t), \quad t^m e^{\alpha t}\sin(\omega t)$$

Where $m \in \mathbb{Z}_{\geq 0}$ and $\mu, \alpha, \beta \in \mathbb{R}$.

If this is the case, then we know the form of the solution. We will assume that g is one of the functions above (if it is linear combination, say $g_1 + g_2$, then if we can solve it for g_1 and g_2, we simply add those solutions to solve the original problem). Let k denote the multiplicity of μ as a root of the characteristic polynomial (so $k = 0$ if μ isn't a root). Then

Theorem 3.1. *If μ is real, then there is a solution of the form*

$$y_*(t) = t^k(\textit{polynomial of degree } m)e^{\mu t}$$

If $\mu = \alpha + i\omega$ is complex, then there is a solution of the form

$$t^k(\textit{poly of deg } m)e^{\alpha t}\cos(\omega t) + t^k(\textit{poly of deg } m)e^{\alpha t}\sin(\omega t)$$

3.0.1. *Example (notes, p. 40, (v)).* Solve

$$y^{(4)} + 4y'' = \sin 2t$$

The characteristic polynomial is $P(\lambda) = \lambda^4 + 4\lambda^2$, which has roots $\lambda_{1,2} = 0$, $\lambda_3 = 2i$, $\lambda_4 = -2i$ (The 0 is repeated).

For $\sin 2t$, we have $m = 0$, $\alpha = 0$, and $\omega = 2$, and $\mu = 2i$ is a root of multiplicity 1, so we get the solution $y = t(A\cos 2t + B\sin 2t)$.

Those constants are *not* for the initial conditions – we can solve for them! For,

$$\begin{aligned} y^{(4)} &= 32(A\sin(2t) - B\cos(2t)) + 16t(A\cos(2t) + B\sin(2t)) \\ &= 16((2A + tB)\sin(2t) + (tA - 2B)\cos(2t)) \end{aligned}$$

and

$$\begin{aligned} 4y'' &= 16(B\cos(2t) - A\sin(2t)) - 16t(A\cos(2t) + B\sin(2t)) \\ &= -16((A + tB)\sin(2t) + (tA - B)\cos(2t)) \end{aligned}$$

So

$$[16((2A + tB)\sin(2t) + (tA - 2B)\cos(2t))] + [-16((A + tB)\sin(2t) + (tA - B)\cos(2t))] = \sin(2t)$$

Which means

$$1 = 16(2A + tB) - 16(A + tB) = 16A$$

and

$$0 = 16(tA - 2B) - 16(tA - B = 16B$$

So $A = 1/16$ and $B = 0$, giving us a final solution of

$$y = \frac{t}{16}\cos(2t)$$

To get the general solution, we would solve the homogenous equation $y^{(4)} + 4y'' = 0$, by observing that the associated polynomial is $k^4 + 4k^2$, which has roots $0, 0, \pm 2i$, so the full solution is:

$$y(t) = \frac{t}{16}\cos(2t) + C_1 + C_2 t + C_3\cos(2t) + C_4\sin(2t)$$

4. Review example - What is a steady state?

You have learned about steady-state solutions, but you may not realize that you did. Recall the Rayleigh equations, which have a limit orbit. The limit orbit is a steady state. You prove that a system has a steady state solution by solving the system (usually by writing it as a second-order differential equation) and showing that what you have is of the form $(x(t), y(t)) = e^{-ct}(f_1(t), f_2(t)) + (s_1(t), s_2(t))$. The initial conditions should only affect f_1 and f_2. Note that one of the terms exponentially dies, and the other one doesn't depend on the initial conditions. The solution $(s_1(t), s_2(t))$ is the steady state solution.

5. Review example - Lyapunov functions

Suppose that $\dot{x} = v(x)$ is a linear system (x is a vector). If you can find a function Φ so that:

(1) Φ has a strict minimum at x^*.
(2) $D_v\Phi(x) < 0$ for all x in some neighborhood of x^*.

Then Φ is a Lyapunov function. The last condition can be replaced by $D_v\Phi(x) \leq 0$ for all x; in that case Φ is a weak Lyapunov function. Note that the v in the directional derivative is the v from the definition of the system.

There are some facts:

- If a system has a strict Lyapunov function at x^*, then the equilibrium is asymptotically stable.
- If a system has a weak Lyapunov function at x^*, then the equilibrium is stable.
- If there is a function which satisfies condition (2) but not (1), then the equilibrium is unstable.

5.1. Example.

Show that the critical point $(0, 0)$ of the system:

$$\begin{cases} \dot{x} &= -y - x \\ \dot{y} &= 2x^3 \end{cases}$$

is stable:

Let's try the function $\Phi(x, y) = ax^4 + by^2$. Then $D_v\Phi(x, y) = (4ax^3, 2by) \cdot (-y - x, 2x^3) = -4ayx^3 - 4ax^4 + 4byx^3$. We'd like to get rid of the yx^3 terms, so let's set $a = b = 1$, Then $D_v\Phi(x, y) = -4x^4$. Note that this is strictly less than 0 for all nonzero x. Also, Φ clearly has a strict minimum at $(0, 0)$. Therefore, the equilibrium at $(0, 0)$ is stable. It is *not* asymptotically stable, because this is just a weak Lyapunov function.

6. Review example - Complex variables

Sometimes, it can help to phrase the problem in terms of complex variables. I know that you haven't really learned about differentiation, etc, but using the method shouldn't be too bad. You can find this on page 64 of the course notes, part 3. Suppose that you have a system $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$ which can be written $\dot{z} = v(z)$ with v analytic (you don't actually have to know complex differentiation—if v is a polynomial, then you differentiate it the same way). You can separate the complex system and solve it.

6.1. **Example.** Solve the system

$$\dot{x} = x^2 - y^2, \quad \dot{y} = 2xy$$

This is just the complex system $\dot{z} = z^2$, which we separate as

$$\frac{1}{z^2} dz = dt$$

We integrate (don't worry about the fact that it's complex) to get

$$\frac{-1}{z} = t + C$$

The constant is complex, but if we take t to be real, then we note that the imaginary part of $\frac{-1}{z}$ must be constant. Well, $\frac{1}{z} = \frac{x-iy}{x^2+y^2}$, so the imaginary part being negative means that $\frac{y}{x^2+y^2}$ is constant for solutions. Therefore the solutions are circles or the positive or negative real line. You can see this by setting $\frac{y}{x^2+y^2} = C$, so $0 = x^2 + y^2 - \frac{1}{C}y$, and completing the square

$$\frac{1}{4C^2} = x^2 + (y - \frac{1}{2c})^2$$

Which is a circle centered at $(0, \frac{1}{2C})$ of radius $\frac{1}{2C}$.

7. Review example - Conservative and dissipative systems

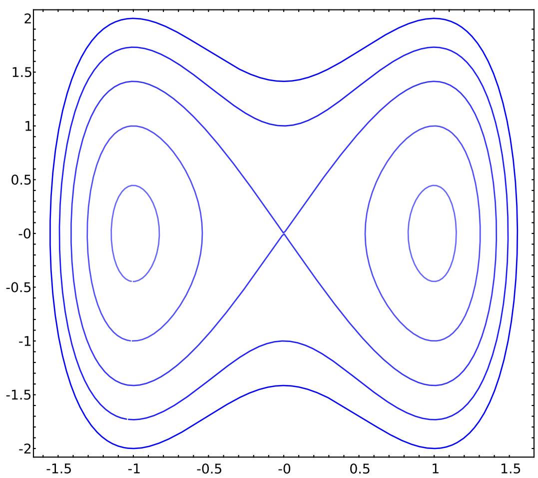
7.1. **Conservative.** Write down the total energy function for a particle of unit mass and a coupled system for x and $y = \dot{x}$ if the potential function is $V(x) = (x-1)^2(x+1)^2 = x^4 - 2x^2 + 1$:

Well, the total energy function is $E(x) = \frac{1}{2}\dot{x}^2 + (x-1)^2(x+1)^2$. We know that $\dot{x}\ddot{x} = -4\dot{x}x^3 + 4\dot{x}x$, so $\dot{y} = -4x^3 + 4x$.

It is not a surprise when you look at the graph of $V(x)$ that the stationary points are $(0, 0)$, $(-1, 0)$, and $(1, 0)$. The matrix of partials is

$$D = \begin{pmatrix} 0 & 1 \\ -12x^2 + 4 & 0 \end{pmatrix}$$

Which, at $(0, 0)$ gives a saddle with eigenvectors $(\pm 1, 2)$ with eigenvalues ± 2. At $(-1, 0)$ and $(1, 0)$ it's the same because of that x^2, and we have eigenvalues $\pm i\sqrt{8}$. Therefore we have centers at those two stationary points. Because the energy is constant, the system must have centers and not spirals. The phase portrait looks like:



If we are interested in finding the approximate periods of small oscillations around the points $x = \pm 1$, then we can apply the note in the class notes on page 67, which says that they have period $2\pi/\omega$, where eigenvalues are $\pm i\omega$. In our case, that would give periods of about $2\pi/\sqrt{8}$.

7.2. **Dissipative.** These are the same, except we introduce a damping term into the energy function, ie now we have $m\ddot{x} = -k\dot{x} - V'(x)$, which gives us a negative derivative of energy with respect to time, and gives a system of the form (for unit mass):

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -ky - V'(x) \end{cases}$$

We study these in the same way, except now the pictures are cooler.

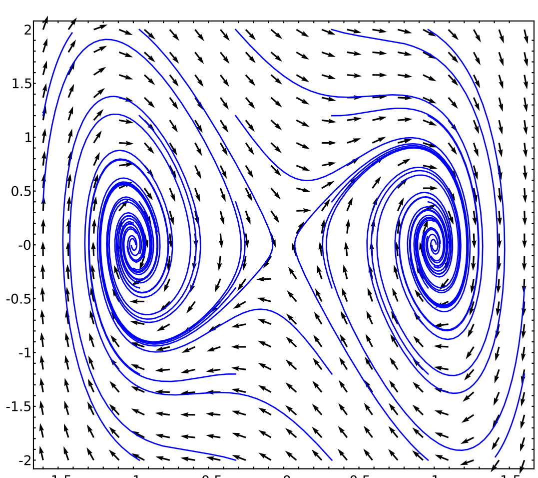
7.3. **Example.** Let's use the system from above but introduce a $-y$ term:

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -y - 4x^3 + 4x \end{cases}$$

The stationary points are the same $((0,0), (\pm 1, 0))$, and the matrix of partials is now

$$D = \begin{pmatrix} 0 & 1 \\ -12x^2 + 4 & -1 \end{pmatrix}$$

For $(0,0)$ this has characteristic polynomial $\lambda^2 + \lambda - 4$, so the eigenvalues are the rather nasty $-1/2 \pm \sqrt{17}/2$, but note this point is still a saddle, so that's good. The other points give eigenvalues of $-1/2 \pm i\sqrt{31}/2$; these points are now stable spirals. The phase portrait now looks like:



8. Review example - Ecological models

Draw a phase portrait for the system

$$\begin{cases} \dot{x} = x(4-2x-2y) \\ \dot{y} = y(9-6x-3y) \end{cases}$$

The stationary points are $(0,0),(2,0),(0,3),(1,1)$. To solve for these points, first try $x=0$ or $y=0$ and get solutions for those. Then assume both are nonzero, divide by them, and get two linear equations in two variables, which you can solve with linear algebra. Next, the matrix of partials is:
$\begin{pmatrix} 4-4x-2y & -2x \\ -6y & 9-6x-6y \end{pmatrix}$
Now let's look at each point:

- $(0,0)$ Here the Jacobian evaluates to $\begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}$, which obviously has eigenvalues 4 and 9 along each axis (an "unstable node").
- $(2,0)$ Here it's $\begin{pmatrix} -4 & -4 \\ 0 & -3 \end{pmatrix}$. This has eigenvalues -4 along the x-axis and -3 with eigenvector $(-4,1)$. It is a "stable node."
- $(0,3)$ Gives $\begin{pmatrix} -2 & 0 \\ -18 & -9 \end{pmatrix}$, which is also a stable node with eigenvalues -2 and -9 for eigenvectors $(7,-18)$ and $(0,1)$.
- $(1,1)$ Gives $\begin{pmatrix} -2 & -2 \\ -6 & -3 \end{pmatrix}$, with eigenvalues -6 and 1 for eigenvectors $(1,2)$ and $(2,-3)$. It is a saddle.

Then, we take these four local pictures and connect them up. Basically, you just draw in a bunch of lines from one to the other that indicate how solution curves will behave. In this case, note that there is an unstable equilibrium at $(1,1)$. If the populations start exactly on the curve passing through that point in the direction of the negative eigenvector, then they will reach equilibrium and survive. If they don't start along that curve, they will head towards one of the stationary points on the axes, so one of the species will become extint.

9. Review example - PDEs

Section 36 (p. 97) in the notes gives you instructions on how to do this problem. The idea is that you

(1) Assume that the solution is of the form $u = X(t)T(t)$ (i.e. it's separable)
(2) Deduce two separate BVPs for X and for T (X has constraints, T does not)
(3) Solve the BVP for X to get the eigenvalues λ_n and eigenfunctions X_n, and plug these in to solve for T for each n (i.e. T_n)
(4) Now you know your solution is $u = \sum c_n X_n T_n$, and the initial conditions on the problem tell you that $\sum c_n X_n = 100$ (a constant).
(5) Using the orthogonality of the X_n, you solve for the c_n, which gives you a solution.

Just follow the notes. You should write out the argument though!

9.1. **Example.** Let's do the string vibration example with the string constant $c=1$ for a string on $[0,\pi]$ which we pluck:

$$u_{xx} = u_{tt}$$

Where $u(0,t) = u(\pi,t) = 0$, $u(x,0) = \sin(x)$ and $u_t(x,0) = 0$ (we pull the string out to the graph of sin and let it go, and it's fixed at the ends). Let's assume that $u(x,t) = X(x)T(t)$. We'll find lots of solutions of this form which satisfy the boundary conditions and then hope that a linear combination satisfies the initial conditions. We know that $u_{xx} = X''(x)T(t) = u_{tt} = X(x)T''(t)$, so we can say that

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = -\lambda$$

And thus $X''(x) + \lambda X(x) = 0$ with the boundary conditions $X(0) = X(\pi) = 0$ (we can assume $T(0) = 1$). The solutions of this system depend on whether λ is positive or negative ($\lambda = 0$ is trivial). If $\lambda < 0$ then the solution is an exponential, which cannot satisfy the boundary conditions. If $\lambda > 0$, then the solution is

$\sin(\sqrt{\lambda}x)$, but we must have $\sqrt{\lambda}$ integral to satisfy the condition $X(\pi) = 0$. Therefore the eigenvalues are $\lambda_n = n^2$ for all $n > 0$ with eigenfunctions $X_n(x) = \sin(nx)$.
Now for each λ_n, we have the equation $T'' + \lambda_n T = 0$, which has general solution

$$T_n(t) = A_n \cos(\sqrt{\lambda_n}t) + B_n \sin(\sqrt{\lambda_n}t) = A_n \cos(nt) + B_n \sin(nt)$$

Each function $X_n T_n$ gives a solution which satisfies the boundary conditions, and any linear combination does too. We now take a sum of them and hope to find A_n and B_n which satisfy the initial conditions. That is, set

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos(nt)\sin(nx) + B_n \sin(nt)\sin(nx)$$

Now $u(x, 0) = \sin(x)$ and $u_t(x, 0) = 0$, which means

$$\sin(x) = \sum_{n=1}^{\infty} A_n \sin(nx) \quad \text{and} \quad 0 = \sum_{n=1}^{\infty} nB_n \sin(nx)$$

We know from the class notes that the solutions are orthogonal, so we can find

$$A_n = \frac{(X_n, f)}{(X_n, X_n)} = \frac{\int_0^\pi \sin(nx)\sin(x)dx}{\int_0^\pi \sin^2(nx)dx} \quad \text{and} \quad B_n = \frac{\int_0^\pi 0 \cdot \sin(nx)dx}{n\int_0^\pi \sin^2(nx)dx}$$

So $A_1 = 1$ and all the other are zero, and $B_n = 0$. Thus

$$u(x,t) = \cos(t)\sin(x)$$

Which does satisfy everything. The reason it worked out so nice is that the initial conditions were so nice (0 and sin). If we have the initial velocity $u_t(x, 0) = 0$, then the B_n will be zero, which is reasonable, but if we had a different "plucking function," then we'd get an infinite series for u. A more realistic function is

$$u(x,0) = f(x) = \begin{cases} x & \text{if } x < \pi/2 \\ \pi - x & \text{if } x \geq \pi/2 \end{cases}$$

Then

$$\begin{aligned} A_n &= \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)dx \\ &= \frac{2}{\pi}\int_0^{\pi/2} x\sin(nx)dx + \frac{2}{\pi}\int_{\pi/2}^{\pi}(\pi - x)\sin(nx)dx \\ &= \frac{2}{\pi}\left.\frac{\sin(nx) - nx\cos(nx)}{n^2}\right|_0^{\pi/2} + \frac{2}{\pi}\left.\frac{\frac{nx\cos(nx)-\sin(nx)}{n} - \pi\cos(nx)}{n}\right|_{\pi/2}^{\pi} \\ &= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{2}{\pi}\frac{2(-1)^{n+1}}{n^2} & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

Don't quote me on those — the point is that the solution $u(x,t) = \sum_{n=1}^{\infty} A_n \cos(nt)\sin(nx)$ is rather nasty, but at least we can solve it!