MATH 2A RECITATION 9/29/11

ALDEN WALKER

1. Front Matter

I am Alden Walker; here is some info:

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Office: Sloan 156
Office Hours: 7–8pm Sunday.

Homework is due **Mondays at noon**. You get two free extensions of however long you want. It would be nice if you could tell me that you are taking an extension before the homework is due so that we don't go looking for it.

You are allowed to use a computer to draw direction fields, but you may not use a computer to do symbolic manipulation for you (like, symbolically solve differential equations).

On my website, on the page for this course, there is a box into which you can enter anonymous comments. If there is some way in which I can do something better, please tell me. If you don't tell me, nothing will change. I don't see tqfr reports from this term until halfway through the winter term, so they are of no immediate value!

2. Differential Equations

Let $x(t)$ be a function, and $\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t}$. We want to find x, given the constraints:

- $F(t, x, \dot{x}, \ddot{x}, \ldots) = 0$
- $x(t_0) = x_0$
- $\dot{x}(t_1) = x_1$
- $\vdots$

The first constraint is the differential equation; the others are the **initial values**, or **boundary values**. When F involves only first derivatives, and we are given that $x(t_0) = x_0$, it is called an **initial value problem** (IVP). We call x_0 the initial value.

2.1. **Example.** Suppose that $\dot{x} = ax$ and $x(0) = 5$. Notice that t never appears, but we understand that x is a function of t. Don't worry about how I came up with this (although you can probably guess), but let's just show that $x(t) = 5e^{at}$ solves the IVP.

First, $\dot{x} = \frac{\mathrm{d}}{\mathrm{d}t}5e^{at} = 5ae^{at} = ax(t) = ax$, so we satisfy the equation, and $x(0) = 5e^{a\cdot 0} = 5$, so that works also. That is all we have to do to verify that a given function solves the IVP.

2.2. **How We Solve Them.** Solving differential equations is hard. Remember how integration is hard? Integration is the "trivial" case for differential equations. We do the following:

- Guessing
- Formulas (remembering other people's guesses)
- Understanding the solution qualitatively when we can't explicitly solve it

2.2.1. *Why is Guessing OK?.* Suppose that we have an IVP, and we manage to guess an answer that works. Is that the only answer? Guessing would be unsatisfactory if we didn't have the following:

Theorem 2.1 (6.2)**.** *If $f(x, t)$ and $\frac{\partial f(x,t)}{\partial x}$ are continuous for $a < x < b$ and $c < t < d$, then for any $x_0 \in (a, b)$ and $t_0 \in (c, d)$, the IVP*

$$\dot{x} = f(x, t) \text{ and } x(t_0) = x_0$$

Has a unique solution on some open interval I containing t_0.

Note that the size of I is not given (the solution is local). All we know is that there is some I containing t_0 on which there is a unique function $x(t)$ which solves the IVP. Even if we manage to solve the IVP, understanding the size of the interval I on which it is defined isn't trivial.

2.3. Three Classes.

Trivial: Solve $\dot{x} = f(t)$. This is "just" integration
Autonomous: Here $\dot{x} = f(x)$. There is no direct dependence of $\dot{x}$ on t.
Separable: $\dot{x} = f(x)g(t)$. We will see these are solvable in a straightforward way

3. Separable

I like to organize recitation so that the topics are covered in the order that they are needed on the homework. That's a little difficult for the first homework since the topics go back and forth, so I have picked this order.

If you need to solve a differential equation of the form $\frac{dx}{dt} = f(x)g(t)$, then you can write $\frac{1}{f(x)}dx = g(t)dt$. If you integrate this, you will obtain an (implicit) solution to the DE. Notice that I've "multiplied by" dt, which of course doesn't make any sense. However, see p.64 for the justification of this trick.

3.1. **Example.** Solve $\dot{x} = tx^2$. Here we have $\frac{1}{x^2}dx = tdt$, so $\int x^{-2}dx = \int tdt$, and we get $-\frac{1}{x} = \frac{t^2}{2} + C$. This is implicit, but we can solve for $x(t) = -\frac{2}{t^2+D}$.

3.2. **Definite Limits.** If we want to put limits on the integrals in this method, we can, but we must be careful—the limits on the x integral are the x limits, and the limits on the t integral are the t limits corresponding to the x limits:

$$\int_{x_0}^{x(t)} \frac{1}{f(x)}dx = \int_{t_0}^{t} g(\tilde{t})d\tilde{t}$$

Notice that you can use this equation to show that integrals might or might not go to infinity.... See page 60.

3.3. **A Helpful Trick.** Suppose that you solve a differential equation which is separable and you obtain an implicit solution which is nasty. Can you say something about the interval I on which the solution is defined? Let's see.

Does the IVP $\dot{x} = x^{1/3}$ and $x(0) = 0$ have a unique solution? Well, we can't use the theorem because $\frac{\partial x^{1/3}}{\partial x}$ isn't continuous at $x = 0$, so let's just try to find two solutions. First, $x(t) = 0$ is a solution. It usually is a solution to autonomous equations. Therefore, let's try to find another solution. Notice that this equation is separable, so we can say:

$$\int \frac{1}{x^{1/3}}dx = \int dt$$

So $\frac{3}{2}x^{2/3} = t$ is an implicit solution. Yes, I know we can solve for x, but let's not. What's another way to see where this solution is defined. Notice that $\frac{\mathrm{d}}{\mathrm{d}x}\frac{3}{2}x^{2/3} = x^{-1/3}$, and $x^{-1/3}$ is strictly positive on $(0, \infty)$. Therefore, $\frac{3}{2}x^{2/3}$ is strictly increasing on $(0, \infty)$, so it is an injective $(1-1)$ function here. Also, it is clearly surjective (onto) because $\lim_{x\to\infty} \frac{3}{2}x^{2/3} = \infty$, and it is 0 at 0. Therefore, $\frac{3}{2}x^{2/3}$ has an inverse function; call it $w(t)$. By the inverse function theorem, it is differentiable.

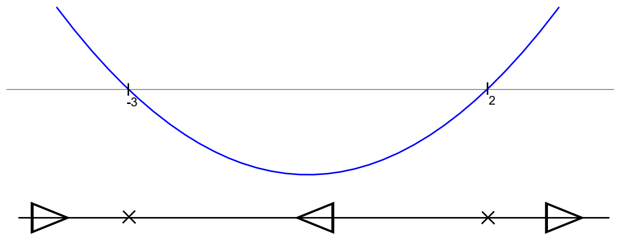
Define $g(t) = \begin{cases} 0 & x \le 0 \\ w(t) & x > 0 \end{cases}$. This gives us a second solution to the differential equation, so we know that the IVP does **not** have a unique solution.

If you were to use a trick like this on your homework, you would also have to show that (a) $g(t)$ is differentiable at 0, probably using a simple limit, (b) $g(t)$ actually solves the differential equation, and (c) $g(t)$ is not identically zero. These aren't hard, but you need them.

4. Qualitative Methods

If you have the autonomous equation $\dot{x} = f(x)$, then you can think of x as the position of a particle. The velocity of the particle depends only on its current position, so you can plot the graph of $f(x)$ and draw arrows which help you see where the particle will move. This is called a **phase diagram**.

4.1. **Example.** Let $\dot{x} = x^2 + x - 5$. I have drawn the phase diagram below. Usually, it is drawn on one line (as I do later) by just combining the plot and the arrows and x's:

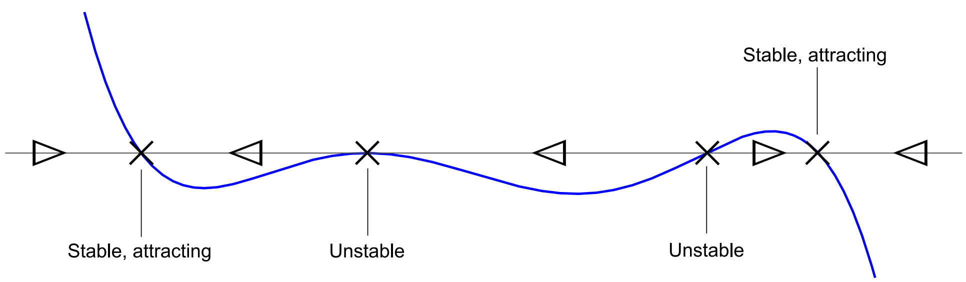


4.2. **Stationary Points.** If there is a point such that $f(x) = 0$, then we call it a **stationary point**, because any particle (solution) which starts there will stay there. A stationary point x^* is:

Stable: if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|x(0) - x^*| < \delta \Rightarrow |x(t) - x^*| < \varepsilon$ for all $t \geq 0$.
Attracting: if there exists $\delta > 0$ such that $|x(0) - x^*| < \delta \Rightarrow \lim_{t\to\infty} x(t) = x^*$.

You will show on your homework that attracting points must be stable. However, it is not the case that stable points must be attracting: consider $f(x) = 0$. Every point is stable but not attracting.

4.2.1. *Example.* Here is a plot of f with the stationary points labeled:



I have used the facts on p.49, which let you classify the stability of points based on f' and some pictures.

5. Hints

There are a few problems requiring proofs on your homework. There is no point in giving you facts or examples here, because you need to find the trick. Here are some possibly helpful hints:

- To prove that a solution to some problem is unique, you assume that there are two solutions x and y, and you prove that $x(t) = y(t)$ for all t (not just $t > 0$), which is the same as proving that $|x(t) - y(t)| = 0$.
- Read p.47 and notice that the solution to an autonomous equation cannot "turn around." This might be useful for problem 5.

5.1. **Example Proof.** Let's prove that if we have a stationary point x^* for the autonomous equation $\dot{x} = f(x)$, and $f'(x^*) < 0$, then x^* is a stable point. I will be assuming that x and f are continuously differentiable.
Because f' is continuous, there is some $c > 0$ such that $f'(x) < 0$ for all $x \in (x^* - c, x^* + c)$. This means that $f(x) > 0$ for all $x \in (x^* - c, x^*)$ and $f(x) < 0$ for $x \in (x^*, x^* + c)$ (I think you know this fact— just apply the mean value theorem to any point for which these don't hold to get a contradiction).
Now, given $\varepsilon > 0$, choose $\delta = \min(c/2, \varepsilon/2)$, and suppose that $|x(0) - x^*| < \delta$. Suppose toward a contradiction that there exists $t' > 0$ such that $|x(t') - x^*| \geq \varepsilon$. Assume that $x(t') > x(0)$ (the proof is essentially the same if $x(t') < x(0)$). By the intermediate value theorem, there exists $t'' \in (0, t')$ such that $x(t'') > x(0)$ and $x(t'') \in (x^*, x^* + c)$. Applying the mean value theorem, there must be a $t''' \in (0, t'')$ such that $\dot{x}(t''') = \frac{x(t'') - x(0)}{t''} > 0$. But then $f(x(t''')) = \dot{x}(t''') > 0$ and $x(t''') \in (x^*, x^* + c)$. This is a contradiction, so we conclude that $|x(t) - x(0)| < \varepsilon$ for all $t > 0$, and x^* is a stable point.