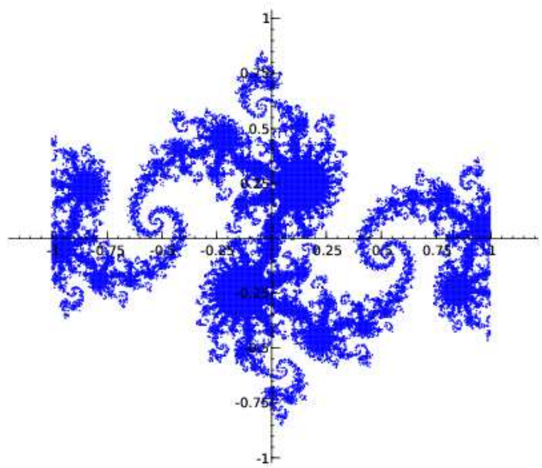
MATH 2A RECITATION 10/13/11

ALDEN WALKER

1. Discrete systems and fractals

You can get fractals in many ways. One of the easiest ways is using a discrete dynamical system, similar to the ones you studied last week. The examples that you looked at were fairly well-behaved. However, if we consider complex valued functions (really, functions of two variables), we can get extremely complicated pictures with very simply-defined systems.

Consider the system $z_n = f(z_{n-1})$ (we think of $z \in \mathbb{C}$, but of course you could write down a 2-d real system if you wanted), where f is a nice function (say, a polynomial), and look at the boundary of the set of points $\{z \mid \lim_{n\to\infty} f^n(z) = \infty\}$. What you get is called a Julia set, which you have probably seen pictures of.



It's not hard to make nice pictures. To get the one above, I just checked Wikipedia to see that $f(z) = z^2 - 0.8 + 0.156i$ gives a nice picture, took 400 x and y values between 0 and 1, and for each pair checked whether $|f^{100}(z)| < 100$. If so, it got colored blue.

You might have heard of the Mandelbrot set. This is not a Julia set of the above form, but the definition feels very similar. It is the set of points c such that $\lim_{n\to\infty} f^n(0) < \infty$, where $f(z) = z^2 + c$. That is, if we vary the input point and keep the polynomial fixed, we get a Julia set, but if we fix the input point (at 0), and vary the polynomial in a family of polynomials, we'll get a set like the Mandelbrot set.

2. Methods of Integration of First Order ODEs

2.1. **Linear ODEs.**
Linear DEs are DEs of the form

$$\frac{dx}{dt} + p(t)x = q(t)$$

And we can solve them, as long as we can do a few integrations.

2.1.1. *How to solve them.* We are going to multiply by an **integrating factor**, that is, something which will allow us to write the left hand side as a single derivative. In this case, the integrating factor is:

$$e^{\int p(t)dt}$$

Giving us the new equation:

$$\begin{aligned} e^{\int p(t)dt}\left(\frac{dx}{dt}+p(t)x\right) &= q(t)e^{\int p(t)dt} \\ \frac{d}{dt}xe^{\int p(t)dt} &= q(t)e^{\int p(t)dt} \end{aligned}$$

Which we can "just" integrate.

2.1.2. *Example.* Solve the DE $\frac{dx}{dt}+tx=4t$, with $x(0)=2$:
Our integrating factor here is $e^{\int t dt}=e^{t^2/2}$, so we rewrite the equation as

$$e^{t^2/2}\left(\frac{dx}{dt}+tx\right)=\frac{d}{dt}xe^{t^2/2}=4te^{t^2/2}$$

So let's integrate both sides with respect to t (although we use a dummy variable s) from the initial value $t_0=0$ to t:

$$x(t)e^{t^2/2}-x(0)=\int_0^t 4se^{s^2/2}ds=\left(4e^{s^2/2}\right|_0^t=4e^{t^2/2}-4$$

Since $x(0)=2$, we have

$$x(t)=4-2e^{-t^2/2}$$

We can check this, and in fact $\frac{dx}{dt}+tx=2te^{-t^2/2}+4t-2te^{-2t^2/2}=4t$, as desired.

2.2. **Exact Equations Again.** We learned about this last week. Recall $f(x,t)dt+g(x,t)dx=0$ is exact iff $\frac{\partial f}{\partial x}=\frac{\partial g}{\partial t}$. We also learned how to solve them: first integrate f with respect to t to get an equation (with an integral) for $E(x,t)$, then differentiate this with respect to x, and you must arrive at $g(x,t)$, allowing you to solve for all the constants.
The problem this week is to find an integrating factor which will make an equation exact. This is hard, and in general cannot be done. However, in some special cases, it can be. Obviously, since otherwise you wouldn't be learning about them.
It turns out that if there is an integrating factor I for the equation, it must satisfy

$$\left(\frac{\partial f}{\partial x}-\frac{\partial g}{\partial t}\right)I=g\frac{\partial I}{\partial t}-f\frac{\partial I}{\partial x}$$

Which doesn't help, unless I is a function only of t, hence:

2.2.1. *How to solve them.* See p.92 in the book.
The equation will have an integrating factor which depends only on t if

$$\frac{1}{g}\left(\frac{\partial f}{\partial x}-\frac{\partial g}{\partial t}\right)$$

Depends only on t, and this integrating factor will satisfy:

$$\frac{dI}{dt}=\frac{1}{g}\left(\frac{\partial f}{\partial x}-\frac{\partial g}{\partial t}\right)I$$

So you solve equations like the above by:

(1) Check if it is exact by checking if $\frac{\partial f}{\partial x}=\frac{\partial g}{\partial t}$
(2) If not, check if the integrating factor depends only on t by checking if $\frac{1}{g}\left(\frac{\partial f}{\partial x}-\frac{\partial g}{\partial t}\right)$ depends only on t
(3) Solve the DE $\frac{dI}{dt}=\frac{1}{g}\left(\frac{\partial f}{\partial x}-\frac{\partial g}{\partial t}\right)I$.
(4) Multiply by I
(5) Check if the equation is exact, which it should be, or assume you're right and don't
(6) Do the algorithm from last week to solve the exact equation.

2.2.2. *Example.* These are hard to make up, so this is from the notes—sorry about that.
Solve $(2x - te^t)dt - tdx = 0$. Well, $\frac{\partial f}{\partial x} = 2$, but $\frac{\partial g}{\partial t} = -1$, so it's not exact. However, we check and see that

$$\frac{1}{g}\left(\frac{\partial f}{\partial x} - \frac{\partial g}{\partial t}\right) = \frac{-1}{t}(2+1) = \frac{-3}{t}$$

So the integrating factor depends only on t. To find it, we solve the differential equation $\frac{dI}{dt} = -(3/t)I$. A solution to this is $I(t) = \frac{1}{t^3}$. I don't know a better way to get that than by guessing. Anyway note that our new equation is:

$$(2(x/t^3) - t^{-2}e^t)dt - t^{-2}dx = 0$$

And we check $\frac{\partial f}{\partial x} = (2/t^3) = \frac{\partial g}{\partial t}$, so it is exact, and you can proceed from here.

2.3. **Homogenous Equations.** Are equations which can be written in the form

$$\frac{dx}{dt} = F\left(\frac{x}{t}\right)$$

Which we care about because then if we set $u = x/t$, then

$$\frac{dx}{dt} = u + t\frac{du}{dt}$$

So

$$t\frac{du}{dt} = F(u) - u$$

Which is separable. The problem about this shouldn't be too hard.

2.4. **Bernoulli Equations.** Bernoulli equations are slight generalizations of linear ODEs: a Bernoulli equation is one of the form:

$$\frac{dx}{dt} + p(t)x = q(t)x^n$$

2.4.1. *How to solve them.* Make the substitution $u = x^{1-n}$, which then gives the differential equation:

$$\begin{aligned}
\frac{du}{dt} &= (1-n)x^{-n}\frac{dx}{dt} \\
&= (1-n)x^{-n}(-p(t)x + q(t)x^n) \\
&= (1-n)(-p(t)x^{1-n} + q(t)) \\
&= -(1-n)p(t)u + (1-n)q(t)
\end{aligned}$$

Which is a linear differential equation. After you solve it (make sure to deal with initial conditions, ie $u_0 = x_0^{1-n}$), then you have the function $u(t)$, so you in turn have the function $x(t) = u^{\frac{1}{1-n}}$.

3. Applications of 1st order ODEs

- You will probably find page 80 to be helpful.
- If a problem doesn't have any derivatives in it, try to find a way to put some derivatives in it. In some cases, the way to do this might be fairly simple. In other cases, you might have to set up a fraction which you can take the limit of (ie the derivative limit fraction).

4. Method Of Linearization

The idea here is that if we have $dx/dt = f(t, x, \varepsilon)$ and $x(t_0) = a(\varepsilon)$, where ε is small, then we can solve the equation assuming that $\varepsilon = 0$, and then extrapolate a little bit for nonzero ε.

4.1. **How to solve it.** First, solve $dx/dt = f(x,t,0)$ with $x(t_0) = a(0)$. Ok no problem! Call $x(t,0) = \varphi(t)$. Then

$$x(t,\varepsilon) \approx x(t,0) + \varepsilon \frac{\partial x}{\partial \varepsilon}(t,0) = \varphi(t) + \varepsilon\psi(t)$$

(We just defined $\psi(t)$), and we want to find $\psi(t)$. It turns out (see p. 23 of the notes) that

$$\frac{d\psi}{dt} = -p(t)\psi(t) + g(t)$$

(a linear ODE) where

$$-p(t) = \frac{\partial f}{\partial x}(t,\varphi(t),0) \text{ and } g(t) = \frac{\partial f}{\partial \varepsilon}(t,\varphi(t),0)$$

And initial condition $\psi(t_0) = a'(0)$.

4.2. **Example.** Solve approximately for $\varphi_\varepsilon(t)$, being the solution to the IVP

$$dx/dt = 2t + (1+\varepsilon)x, \text{ with } x(0) = \varepsilon$$

First, we solve the IVP $dx/dt = 2t + x$ with $x(0) = 0$. This is a linear ODE and it has integrating factor e^{-t}, so we have

$$x(t)e^{-t} - x(0) = 2\int te^{-t}dt = -2(te^{-t} + e^{-t} - 1)$$

so $\varphi(t) = x(t) = -2(t + 1 - e^t) = 2(e^t - t - 1)$. Nice. Next, we set up the linear ODE

$$d\psi/dt = -p(t)\psi(t) + g(t)$$

Where

$$-p(t) = \frac{\partial f}{\partial x}(t,\varphi(t),0) = 1$$

and

$$g(t) = \frac{\partial f}{\partial \varepsilon}(t,\varphi(t),0) = \varphi(t) = 2(e^t - t - 1)$$

And $\psi(0) = a'(0) = 1$. Ie $d\psi/dt = \psi(t) + 2(e^t - t - 1)$, or $d\psi/dt - \psi(t) = 2(e^t - t - 1)$, so the integrating factor is $e^{-\int 1dt} = e^{-t}$, and we are solving

$$\frac{d}{dt}\psi(t)e^{-t} = 2e^{-t}(e^t - t - 1) = 2 - 2te^{-t} - 2e^{-t}$$

ie

$$\begin{aligned}
\psi(t)e^{-t} - \psi(0) &= 2\int_0^t ds - 2\int_0^t se^{-s}ds - 2\int_0^t e^{-s}ds \\
\psi(t)e^{-t} - 1 &= 2t - 2\left(-se^{-s} - e^{-s}\right|_0^t - 2\left(-e^{-s}\right|_0^t \\
&= 2t - 2(-te^{-t} - e^{-t} + 1) - 2(-e^{-t} + 1) \\
&= 2(t+2)e^{-t} + 2t - 4
\end{aligned}$$

So we get

$$\psi(t) = 2(t+2) + (2t-3)e^t$$

So finally,

$$\begin{aligned}
\varphi_\varepsilon(t) &= \varphi(t) + \varepsilon\psi(t) \\
&= 2(e^t - t - 1) + \varepsilon(2(t+2) + (2t-3)e^t)
\end{aligned}$$