MATH 2A RECITATION 10/20/11

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The midterm will be handed out 10/24, and it will be due 10/31. There will be a review session (details to follow).

1. Random Thoughts

Let f_n be the n^{th} Fibonacci number, where $f_0 = f_1 = 1$. Is there a closed-form expression for the$n$th Fibonacci number? Yes there is!

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_n \\ f_{n-1} \end{pmatrix} = \begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix}$$

So $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} f_n \\ f_{n-1} \end{pmatrix}$

If you find the eigenvalues of that matrix (call it A), you will see that they are $\varphi = \frac{1+\sqrt{5}}{2}$ and $1-\varphi$, and that the corresponding eigenvectors are $\begin{pmatrix} \varphi \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1-\varphi \\ 1 \end{pmatrix}$, and thus

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \varphi & 1-\varphi \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{5-\sqrt{5}}{10} \\ \frac{-1}{\sqrt{5}} & \frac{5+\sqrt{5}}{10} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

And if you multiply that out and take the bottom coordinate, we know you will get f_n. It comes out to $f_n = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}}$. Pretty cool!

2. Integration of 1st Order Equations

You already have all the techniques for these. One thing I want to mention is that you know how to find integrating factors for (non) exact equations, and you have seen these formulas in terms of, say, finding an integration factor depending only on x. Note that you can do exactly the same thing with the variables swapped, and everything still works. Sometimes you won't be able to find an integrating factor depending only on x, but you will be able to find one depending only on t.

3. 2D Autonomous Rystems and Reducible 2nd Order Equations

Sometimes, you can do tricks to solve equations with higher order derivatives; for instance, if you can multiply by something to make the left hand side a derivative, then you can integrate out the second derivative.

Example. Solve $y''y/y' + y' = 2/y'$. Note that we can multiple both sides by y' to get $y''y + (y')^2 = 2$, and the left hand side is $d/dx(y'y)$, so we see $y'y = 2t + C$, so we separate and integrate to get $y^2/2 = t^2 + ct$, or $y = \pm\sqrt{2t^2 + ct}$. On homework, you should then check this. The thing you multiply by on your homework is not y—you have to fiddle with it until you get the right factor.

3.1. **Change of Variables.**

As discussed in class, if you have a diffeomorphism between two spaces, then a solution to a differential equation is mapped to a solution of a corresponding differential equation. The notion of "corresponding" differential equation can be messy, so the best sorts of variable changes are things like rotations (isometries), where the Jacobian has determinant ± 1. This works best with word problems (hint hint), where you can simply choose to write down equations for the problem in terms of a different orientation, so that vectors point in a coordinate direction.

4. 2nd Order Linear Equations with Constant Coefficients

4.1. **General Theory.** 2nd order linear ODEs are equations of the form:

$$\ddot{x} + f(t)\dot{x} + g(t)x = h(t)$$

They are called **homogeneous** if $h(t) = 0$.

This probably the first real application of linear algebra in the curriculum which is a little weird. I will talk about the case where you only have derivatives of order 2, but this applies in general. Consider the map $L : C^2(I) \to C(I)$ (and probably both are C^∞) defined by $L(x) = \ddot{x} + f(t)\dot{x} + g(t)x$. This is a linear map between vector spaces. Refer to page 33 of the (first) notes. The key is to consider the kernel $\ker L$. Note that the kernel is the set of all solutions to the homogeneous equation. Because $C^n(I)$ is infinite dimensional, this linear algebra is not easy.

Fact: $\ker L$ is of dimension 2. This means that to specify a general solution to the homogeneous equation, you need to provide two linearly independent solutions x_1 and x_2, and then the general solution is $x(t) = Ax_1(t) + Bx_2(t)$, where A and B range over $\mathbb{R}$. In addition, still thinking with linear algebra, if you want a general solution to the nonhomogeneous equation $L(x) = h(t)$, you can find a particular solution x^*, and then the general solution is of the form $x = x^* + Ax_1 + Bx_2$, where x_1, x_2, A, B are as before.

Once you have the general solution, you have to specify two things, such as $x(0)$ and $\dot{x}(0)$, to solve for both constants to get the specific solution you desire.

4.2. **Homogeneous!** These are equations of the form $a\ddot{x} + b\dot{x} + cx = 0$. In this case, a solution is something like e^{kt}, and you just have to find out what k is. You will find k (see how below) by solving a quadratic equation, so there are a few possibilities:

- If you get 2 distinct real roots, then those give you two linearly independent solutions
- If you get a single repeated real root, then the solutions are e^{kt} and te^{kt}
- If you get complex conjugates $k = \alpha \pm i\beta$, then the solution is $e^{\alpha t}(A\cos(\beta t) + B\sin(\beta t))$. Note it's *the* solution, because it gives a subspace of dimension 2 in C^2.

4.2.1. *How to Solve it.* To find k, simply plug in e^{kt} to get $ak^2e^{kt} + bke^{kt} + ce^{kt} = 0$, and divide by e^{kt} and solve the quadratic $ak^2 + bk + c = 0$.

4.2.2. *Example.* Solve $\ddot{x} - \dot{x} - 1 = 0$ where $x(0) = 1$ and $\dot{x}(0) = 1$. We plug in e^{kt} and divide by e^{kt} to get the equation $k^2 - k - 1 = 0$, which has solutions φ and $1 - \varphi$ (ie $\frac{1\pm\sqrt{5}}{2}$), so the general solution is $x(t) = Ae^{\varphi t} + Be^{(1-\varphi)t}$. To solve for A and B, we note the initial conditions give us $A + B = 1$ and $A\varphi + B(1-\varphi) = 1$. Then we can use linear algebra (row reduce the matrix $\left(\begin{array}{cc|c} 1 & 1 & 1 \\ \varphi & 1-\varphi & 1 \end{array}\right)$ or invert it) to get $A = \varphi/\sqrt{5}$ and $B = (\varphi - 1)/\sqrt{5}$.

4.3. **Reduction of Order.** Now we start dealing with nonhomogeneous linear ODEs. We can use the reduction of order technique to simplify the situation. You should read chapter 14 in the book—it has instructions for what do if the right hand side of the equation is a polynomial, exponential, etc, so on a problem or two you might be able to just follow those instructions, or you can try this.

With this strategy, we want to find a solution to $L(x) = f(t)$. If we take a solution y to the homogeneous $L(y) = 0$ and try to find a solution $v(t)$ to the nonhomogeneous $L(yv) = f(t)$, we'll find that the order of this equation is one less (for v').

Also, if you know one solution to a homogeneous equation, this method will produce a second, linearly independent solution! (see chapter 17).

4.3.1. *Example.* Solve $\ddot{x} - 2\dot{x} = t$, with $x(0) = 1$, $\dot{x}(0) = 2$. We solve the homogeneous system to get $y(t) = A + Be^{2t}$ (this is example 12.2), so here $A + B = 1$ and $A + 2B = 2$, so $y(t) = e^{2t}$. Next, we look for a solution v to

$$\begin{aligned} t &= \frac{d^2}{dt^2}(yv) - 2\frac{d}{dt}(yv) \\ &= (y''v + y'v') + (v'y' + v''y) - 2(y'v + v'y) \\ &= (y'' - 2y')v + yv'' + 2(y' - y)v' \end{aligned}$$

So we want to solve $yv'' + 2(y' - y)v' = t$ for v'. This is just a first order linear ODE. To complete the example, call $u = v'$, so we want to solve $e^{2t}u' + 2e^{2t}u = t$. Divide by the exponential, which is never zero, to get $u' + 2u = te^{-2t}$. The integrating factor is e^{2t}, so we integrate to get $e^{2t}u(t) - u(0) = \int_0^t s ds = t^2/2$, so $u(t) = (1/2)t^2e^{-2t}$. Since $yv = x$, we have $y'v + v'y = x'$, so $2 = 2v + v'$, and $yv = x$, so $1v = 1$, so $2 = 2 + v'$, so $u(0) = v'(0) = 0$. Then we solve

$$v(t) = \int_0^t (1/2)s^2e^{-2s}ds = 1/8e^{-2t}(-1 + e^{2t} - 2t - 2t^2)$$

Which we multiply by y to get $x = 1/8(-1 + e^{2t} - 2t - 2t^2)$. This actually works! Make sure you keeps everything straight—this example took like 1.5 hours because I kept making silly mistakes.

4.4. **Variation of Constants.** This method is the most general: we are solving

$$\ddot{x} + p(t)\dot{x} + q(t)x = g(t)$$

We will use solutions to the homogeneous equation to get at these.

4.4.1. *How to Solve Them.* The idea is that if we have two solutions to the homogeneous equation x_1 and x_2, then we will try to find a solution to the nonhomogeneous equation of the form

$$x(t) = u_1(t)x_1(t) + u_2(t)x_2(t)$$

Ie I guess we're letting the constants vary, which means they're not constants. We will solve for u_1 and u_2, but we only have a single equation constraint, so we can actually impose an additional constraint to make our life easier: we assume also that

$$\dot{u_1}(t)x_1(t) + \dot{u_2}(t)x_2(t) = 0$$

Which will simplify the derivative of x, so

$$\dot{x}(t) = u_1(t)\dot{x_1}(t) + u_2\dot{x_2}(t)$$

If we plug this into the equation and simplify a little (see chapter 18 for all of this), then we see that

$$\dot{u_1}(t) = -\frac{x_2(t)g(t)}{x_1(t)\dot{x_2}(t) - x_2(t)\dot{x_1}(t)}$$

and

$$\dot{u_2}(t) = \frac{x_1(t)g(t)}{x_1(t)\dot{x_2}(t) - x_2(t)\dot{x_1}(t)}$$

So then we just integrate, and bam, no problem.

4.4.2. *Example.* We'll do example 18.1 from the book, because it is nice and the answer will be correct. Solve:

$$\ddot{x} + \dot{x} - 6x = 5e^{-3t}$$

First, two solutions to the nonhomogeneous equation are $x_1 = e^{-3t}$ and $x_2 = e^{2t}$ since we have e^{kt} where $k^2 + k + 6 = (k - 2)(k + 3) = 0$. Ok the denominator of those fractions is $x_1(t)\dot{x_2}(t) - x_2(t)\dot{x_1}(t) = 2e^{-3t}e^{2t} + 3e^{2t}e^{-3t} = 5e^{-t}$, so we integrate to find

$$u_1(t) = -\int_0^t \frac{e^{2s}5e^{-3s}}{5e^{-s}}ds = -t$$

and

$$u_2(t) = \int_0^t \frac{e^{-3s}5e^{-3s}}{5e^{-s}}ds = \int_0^t e^{-5s}ds = (-1/5)e^{-5t}$$

So

$$x(t) = u_1(t)x_1(t) + u_2(t)x_2(t) = -te^{-3t} - (1/5)e^{-3t}$$

Nice! Actually we can ignore the second term here because it is a solution to the homogeneous equation.

5. 2nd Order Difference Equations

These are the same as regular difference equations, except the next value in the sequence might depend on more than just the last value. A good example is, again the Fibonacci sequence $x_n = x_{n-1} + x_{n-2}$, $x_0 = 1$, $x_1 = 1$. Solving these is pretty much the same as for second order differential equations

5.1. **How to Solve Them.** Guess the solution to a homogeneous equation $ax_{n+2} + bx_{n+1} + cx_n = 0$ as k^n, and plug in. You then get a quadratic in k.

- If you get 2 distinct real roots, then those give you two linearly independent solutions
- If you get a single repeated real root, then the solutions are k^n and nk^n.
- If you get complex roots, see section 22.3.3 (it's basically the same as before)

5.2. **Example.** Here's maybe even a faster way to get the closed-form version of the Fibonacci sequence: The equation is $x_{n+2} - x_{n+1} - x_n = 0$, so we solve the polynomial $k^2 - k - 1 = 0$, getting roots φ and $(1 - \varphi)$, so the general solution is $x_n = A\varphi^n + B(1 - \varphi)^n$. The initial conditions $x_0 = 1$, $x_1 = 1$ require that $A + B = 1$ and $A\varphi + B(1 - \varphi) = 1$. We already solved this—$A = \varphi/\sqrt{5}$ and $B = (\varphi - 1)/\sqrt{5}$, so $x_n = (1/\sqrt{5})(\varphi^{n+1} - (1 - \varphi)^{n+1})$. The $n + 1$ just comes from indexing starting at 0.