MATH 2A MIDTERM REVIEW 10/27/11

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1. Info

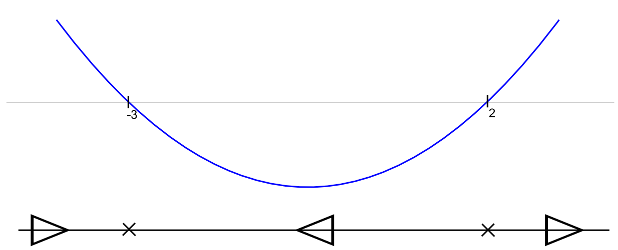
The midterm is due Monday, 10/31. It is open book, open note, open recitation notes, etc. Not open google, mathoverflow, etc.

2. Things to know about

(1) (Local) existence and uniqueness of solutions
(2) Qualitative methods:
  (a) For autonomous equations (phase diagrams)
  (b) For discrete systems (cobweb diagrams)
  (c) For any ODE (slope field)
(3) Approximation of solutions:
  (a) Euler's method
  (b) Picard's method
  (c) How to take limits to sometimes solve with these methods
(4) Separable/exact, etc
  (a) Trick for separable
  (b) Testing for exactness and using exactness to find a solution
  (c) Finding an integrating factor for equations which are not exact
(5) Random tricks:
  (a) Homogeneous equations (as in, $F(x/y)$, not as in linear diffeqs)
  (b) Bernoullli equations
  (c) Linear substitution
(6) Method of linearization
(7) Linear differential equations
  (a) First order linear equations ($\dot{x} + p(t)\dot{x} = q(t)$) (how to always solve them)
  (b) Second order equations
    (i) How to always solve homogeneous second order linear equations with constant coefficients
    (ii) How to go from a solution to the homogeneous equation to a general solution:
      (A) Reduction of order
      (B) Variation of constants
    (iii) How to solve a second order linear equation with *arbitrary* coefficients if you have a solution to the homogeneous equation (e.g. reduction of order, etc)

3. Review Example 1

Let $\dot{x} = x^2 + x - 5$. I have drawn the phase diagram below. Usually, it is drawn on one line (as I do later) by just combining the plot and the arrows and x's:

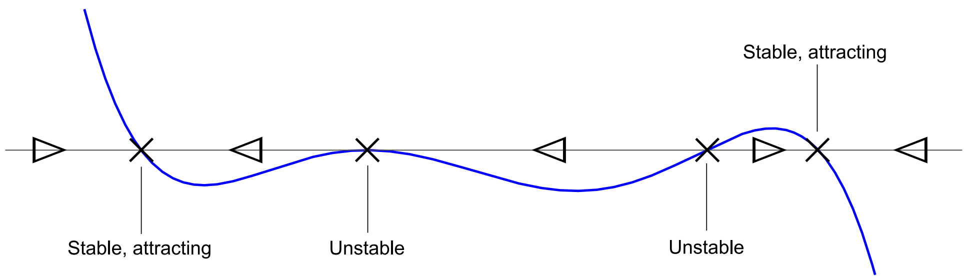


3.1. **Stationary Points.** If there is a point such that $f(x) = 0$, then we call it a **stationary point**, because any particle (solution) which starts there will stay there. A stationary point x^* is:

Stable: if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|x(0) - x^*| < \delta \Rightarrow |x(t) - x^*| < \varepsilon$ for all $t \geq 0$.
Attracting: if there exists $\delta > 0$ such that $|x(0) - x^*| < \delta \Rightarrow \lim_{t\to\infty} x(t) = x^*$.

Attracting points must be stable. However, it is not the case that stable points must be attracting: consider $f(x) = 0$. Every point is stable but not attracting.

3.1.1. *Example.* Here is a phase diagram for some autonomous equation $\dot{x} = f(x)$



I have used the facts on p.49, which let you classify the stability of points based on f' and some pictures.

4. Review example 2

Describe the behavior of $x_{n+1} = \sin(\frac{\pi}{2}x_n)$ for $x_0 \in (0, 1)$. What is the limit of the sequence

$$\sin(\frac{\pi}{2}\sin(\frac{\pi}{2}\sin(\frac{\pi}{2}\cdots\frac{1}{2})))\cdots)$$

Well, the function $x = \sin(\frac{\pi}{2}x)$ has zeros at 0 and 1, so we see that on the interval $(0, 1)$, $\sin(\frac{\pi}{2}x) \geq x$. Therefore, if $x_0 \in (0, 1)$, the sequence x_n is monotone. In addition, it is bounded above because $\sin(\frac{\pi}{2}x) \leq 1$. Therefore, this sequence has a limit. Call this limit L. Clearly,

$$\sin^{-1}(L) = \frac{\pi}{2}L$$

I.e. we have $L = \sin(\frac{\pi}{2}L)$, so L is 0 or 1. However, since the sequence is increasing, if we start in $(0, 1)$, the limit must be 1. The starting point doesn't matter for this analysis, so a special case is the limit requested. **Note:** We did not study all the orbits. If you wanted to study all the orbits, you would need to consider starting points outside $(0, 1)$.

5. Review example 3

Let's think about exact equations.
Since $E_t = M$, we have

$$E(x, t) = \int_{t_0}^{t} M(x, s)\, ds + h(x)$$

We then differentiate with respect to x, and the fact that $E_x = N$ gives us

$$N(x, t) = \int_{t_0}^{t} M_x(x, s)\, ds + h'(x)$$

Which will allow us to find h, assuming we can antidifferentiate.
We want to solve the equation:

$$(2x^2 + t)dt + (4xt + 1)dx = 0$$

We check that it's exact by first noting that it's $\mathcal{C}^1$ and defined everywhere (a rectangle), and that $\frac{\partial}{\partial x}(2x^2+t) = 4x = \frac{\partial}{\partial t}(4xt+1)$. Therefore it is exact, and we can hope to recover a solution E (a **first integral**). Following the method,

$$\begin{aligned} E(x,t) &= \int_{t_0}^{t} (2x^2+s)\,ds + h(x) \\ &= 2x^2 t + t^2/2 + h(x) \end{aligned}$$

If you are distressed by the apparent disappearance of the t_0 term in the integral, have no fear, it is safely thought of as part of $h(x)$ since it is constant with respect to t.
Then we differentiate, since we know $E_x = N$, ie

$$4xt + 1 = N(x,t) = E_x = 4xt + h'(x)$$

So $h'(x) = 1$, so $h(x) = x + C$. Therefore our solution is $E(x,t) = 2x^2t + t^2/2 + x = C$.
If we want to get a specific solution, we need to plug in given values to solve for C.

6. Review example 4

If your equation is not exact, but you suspect it is close:

(1) Check if it is exact by checking if $\frac{\partial f}{\partial x} = \frac{\partial g}{\partial t}$
(2) If not, check if the integrating factor depends only on t by checking if $\frac{1}{g}\left(\frac{\partial f}{\partial x} - \frac{\partial g}{\partial t}\right)$ depends only on t
(3) Solve the DE $\frac{dI}{dt} = \frac{1}{g}\left(\frac{\partial f}{\partial x} - \frac{\partial g}{\partial t}\right) I$.
(4) Multiply by I
(5) Check if the equation is exact, which it should be, or assume you're right and don't
(6) Do the algorithm which solve exact equations

These are hard to make up, so this is from the notes—sorry about that.
Solve $(2x - te^t)dt - tdx = 0$. Well, $\frac{\partial f}{\partial x} = 2$, but $\frac{\partial g}{\partial t} = -1$, so it's not exact. However, we check and see that

$$\frac{1}{g}\left(\frac{\partial f}{\partial x} - \frac{\partial g}{\partial t}\right) = \frac{-1}{t}(2+1) = \frac{-3}{t}$$

So the integrating factor depends only on t. To find it, we solve the differential equation $\frac{dI}{dt} = -(3/t)I$. A solution to this is $I(t) = \frac{1}{t^3}$. I don't know a better way to get that than by guessing. Anyway note that our new equation is:

$$(2(x/t^3) - t^{-2}e^t)dt - t^{-2}dx = 0$$

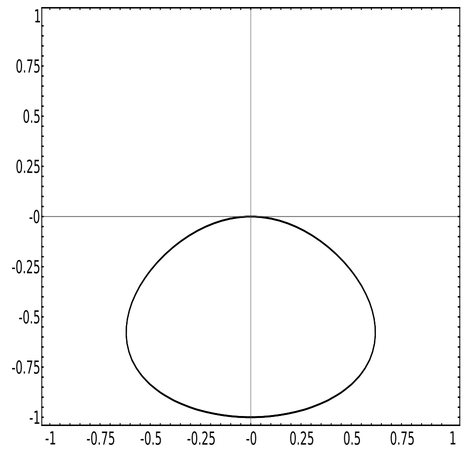
And we check $\frac{\partial f}{\partial x} = (2/t^3) = \frac{\partial g}{\partial t}$, so it is exact, and you can proceed from here.

7. Review example 5

In order to determine where a solution to an exact equation is defined, you solve the equation to get an implicit solution. Using initial conditions, you know what level curve of the first integral gives the particular solution. Now you have to think. Probably, the level curve is a circle or something (not a function). Therefore, the solution is only defined on part of the possible domain. Based on where the initial point is, you figure out how far you can go before something becomes undefined. Usually, you just have to figure out where the derivative goes to infinity.
Solve $\frac{dx}{dt} = \frac{2t}{3x^2-1}$ with $x(0) = 0$: We separate to get $(3x^2-1)dx = (2t)dt$, and integrating,

$$x^3 - x = t^2$$

In this case, when $t = 0$, we have $x = 0$, so $C = 0$. Here is an implicit plot of this equation:



Note that the solution to the IVP is *not* the whole plot: since the graph doubles back, the solution is only defined between the vertical tangents. We can determine this algebraically, since if we look at the differential equation, we see that the solution will only be defined where $3x^2 - 1 \neq 0$, that is, since we know $x(0) = 0$ (so $3x^2 - 1 < 0$), we see we must have $-1/\sqrt{3} < x < 1/\sqrt{3}$, or rather, $-\sqrt{3^{-1/2} - 3^{-3/2}} < t < \sqrt{3^{-1/2} - 3^{-3/2}} \approx 0.62$.

Note: You do **not** always have to solve an equation in order to find its domain. You can often just look at the equation and determine the qualitative behavior and observe when the derivative goes to infinity, etc.

8. Review example 6 (Change of variables)

Sometimes, it's helpful to do a change of variables in order to solve a problem. You did this with the goose problem on your homework. Let's see another one. Suppose we have an equation

$$(2x + 5t)\dot{x} = x + 2t + 1$$

(i.e. something with linear functions of x and t on each side). What you want to do with these is figure out where the point of intersection of the lines is, and do the linear substitution that brings it to the origin. Here we have $2x + 5t = 0$ and $x + 2t + 1 = 0$. Using linear algebra, we find that the point of intersection is $x = -5$, $t = 2$. Thus, do the substitution $u = x + 5$, $v = t - 2$. Now we have

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 1 \qquad \frac{\mathrm{d}v}{\mathrm{d}t} = 1$$

So

$$\frac{\mathrm{d}u}{\mathrm{d}v} = \frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}v} = \frac{\mathrm{d}x}{\mathrm{d}t}$$

So

$$(2(u-5) + 5(v+2))\frac{\mathrm{d}u}{\mathrm{d}v} = (u-5) + 2(v+2) + 1$$

or

$$(2u + 5v)\frac{\mathrm{d}u}{\mathrm{d}v} = u + 2v$$

So we've gotten rid of that 1!. Now to finish it off, do the substitution $z = u/v$. So $\frac{\mathrm{d}u}{\mathrm{d}v} = v\frac{\mathrm{d}z}{\mathrm{d}v} + z$, so

$$(2u + 5v)\left(v\frac{\mathrm{d}z}{\mathrm{d}v} + z\right) = u + 2v$$

and

$$(2\frac{u}{v} + 5)\left(v\frac{\mathrm{d}z}{\mathrm{d}v} + z\right) = \frac{u}{v} + 2$$

or

$$(2z + 5)\left(v\frac{\mathrm{d}z}{\mathrm{d}v} + z\right) = z + 2$$

Rearranging,

$$v\frac{\mathrm{d}z}{\mathrm{d}v} = \frac{z+2}{2z+5} - z$$

Which is separable, so we can just integrate it.

9. Review example 7 (linear DEs)

9.1. **Constant coefficients.** These are equations of the form $a\ddot{x} + b\dot{x} + cx = 0$. In this case, a solution is something like e^{kt}, and you just have to find out what k is. You will find k (see how below) by solving the quadratic equation $ak^2 + bk + c = 0$; there are a few possibilities:

- If you get 2 distinct real roots, then those give you two linearly independent solutions
- If you get a single repeated real root, then the solutions are e^{kt} and te^{kt}
- If you get complex conjugates $k = \alpha \pm i\beta$, then the solution is $e^{\alpha t}(A\cos(\beta t) + B\sin(\beta t))$. Note it's *the* solution, because it gives a subspace of dimension 2 in C^2.

9.1.1. *Example.* Solve $\ddot{x} - \dot{x} - 1 = 0$ where $x(0) = 1$ and $\dot{x}(0) = 1$. We plug in e^{kt} and divide by e^{kt} to get the equation $k^2 - k - 1 = 0$, which has solutions φ and $1 - \varphi$ (ie $\frac{1\pm\sqrt{5}}{2}$), so the general solution is $x(t) = Ae^{\varphi t} + Be^{(1-\varphi)t}$. To solve for A and B, we note the initial conditions give us $A + B = 1$ and $A\varphi + B(1-\varphi) = 1$. Then we can use linear algebra (row reduce the matrix $\left(\begin{array}{cc|c} 1 & 1 & 1 \\ \varphi & 1-\varphi & 1 \end{array}\right)$ or invert it) to get $A = \varphi/\sqrt{5}$ and $B = (\varphi - 1)/\sqrt{5}$.

9.2. **Reduction of Order.** With this strategy, we want to find a solution to $L(x) = f(t)$. If we take a solution y to the homogeneous $L(y) = 0$ and try to find a solution $v(t)$ to the nonhomogeneous $L(yv) = f(t)$, we'll find that the order of this equation is one less (for v').

Awesome fact: If you know one solution to a homogeneous equation, this method will produce a second, linearly independent solution! (see chapter 17).

9.2.1. *Example.* Solve $\ddot{x} - 2\dot{x} = t$, with $x(0) = 1$, $\dot{x}(0) = 2$. We solve the homogeneous system to get $y(t) = A + Be^{2t}$ (this is example 12.2), so here $A + B = 1$ and $A + 2B = 2$, so $y(t) = e^{2t}$. Next, we look for a solution v to

$$\begin{aligned} t &= \frac{d^2}{dt^2}(yv) - 2\frac{d}{dt}(yv) \\ &= (y''v + y'v') + (v'y' + v''y) - 2(y'v + v'y) \\ &= (y'' - 2y')v + yv'' + 2(y' - y)v' \end{aligned}$$

So we want to solve $yv'' + 2(y' - y)v' = t$ for v'. This is just a first order linear ODE. To complete the example, call $u = v'$, so we want to solve $e^{2t}u' + 2e^{2t}u = t$. Divide by the exponential, which is never zero, to get $u' + 2u = te^{-2t}$. The integrating factor is e^{2t}, so we integrate to get $e^{2t}u(t) - u(0) = \int_0^t s ds = t^2/2$, so $u(t) = (1/2)t^2e^{-2t}$. Since $yv = x$, we have $y'v + v'y = x'$, so $2 = 2v + v'$, and $yv = x$, so $1v = 1$, so $2 = 2 + v'$, so $u(0) = v'(0) = 0$. Then we solve

$$v(t) = \int_0^t (1/2)s^2e^{-2s}ds = 1/8e^{-2t}(-1 + e^{2t} - 2t - 2t^2)$$

Which we multiply by y to get $x = 1/8(-1 + e^{2t} - 2t - 2t^2)$. This actually works! Make sure you keeps everything straight—this example took like 1.5 hours because I kept making silly mistakes.

9.3. **Variation of Constants.** This method is the most general: we are solving

$$\ddot{x} + p(t)\dot{x} + q(t)x = g(t)$$

We'll do example 18.1 from the book, because it is nice and the answer will be correct. Solve:

$$\ddot{x} + \dot{x} - 6x = 5e^{-3t}$$

First, two solutions to the nonhomogeneous equation are $x_1 = e^{-3t}$ and $x_2 = e^{2t}$ since we have e^{kt} where $k^2 + k + 6 = (k-2)(k+3) = 0$. Ok the denominator of those fractions is $x_1(t)\dot{x_2}(t) - x_2(t)\dot{x_1}(t) = 2e^{-3t}e^{2t} + 3e^{2t}e^{-3t} = 5e^{-t}$, so we integrate to find

$$u_1(t) = -\int_0^t \frac{e^{2s}5e^{-3s}}{5e^{-s}} ds = -t$$

and

$$u_2(t) = \int_0^t \frac{e^{-3s}5e^{-3s}}{5e^{-s}} ds = \int_0^t e^{-5s} ds = (-1/5)e^{-5t}$$

So

$$x(t) = u_1(t)x_1(t) + u_2(t)x_2(t) = -te^{-3t} - (1/5)e^{-3t}$$

Nice! Actually we can ignore the second term here because it is a solution to the homogeneous equation.