MATH 2A RECITATION 11/3/11

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1. Dragging vectors around the sphere

Have you ever noticed that if you drag a vector around on the sphere, you can get any other vector? Pretty cool, huh?

2. Undetermined Coefficients (non homogeneous)

If the equation $L[y] = g$ has constant coefficients, and g is a quasi polynomial, then we can write the form of a particular solution. Here, quasi polynomial means that g is a linear combination of functions of the form:

$$t^m e^{\mu t}, \quad t^m e^{\alpha t}\cos(\omega t), \quad t^m e^{\alpha t}\sin(\omega t)$$

Where $m \in \mathbb{Z}_{\geq 0}$ and $\mu, \alpha, \beta \in \mathbb{R}$.

If this is the case, then we know the form of the solution. We will assume that g is one of the functions above (if it is linear combination, say $g_1 + g_2$, then if we can solve it for g_1 and g_2, we simply add those solutions to solve the original problem). Let k denote the multiplicity of μ as a root of the characteristic polynomial (so $k = 0$ if μ isn't a root). Then

Theorem 2.1. *If μ is real, then there is a solution of the form*

$$y_*(t) = t^k(\text{polynomial of degree } m)e^{\mu t}$$

If $\mu = \alpha + i\omega$ is complex, then there is a solution of the form

$$t^k(\text{poly of deg } m)e^{\alpha t}\cos(\omega t) + t^k(\text{poly of deg } m)e^{\alpha t}\sin(\omega t)$$

2.0.1. *Example (notes, p. 40, (v)).* Get a general form for a solution to

$$y^{(4)} + 4y'' = \sin 2t + te^t + 4$$

The characteristic polynomial is $P(\lambda) = \lambda^4 + 4\lambda^2$, which has roots $\lambda_{1,2} = 0$, $\lambda_3 = 2i$, $\lambda_4 = -2i$ (The 0 is repeated). We split the quasi polynomial on the right into $g_1 + g_2 + g_3$, where $g_1 = \sin 2t$, $g_2 = te^t$, and $g_3 = 4$.

For g_1, we have $m = 0$, $\alpha = 0$, and $\omega = 2$, and $\mu = 2i$ is a root of multiplicity 1, so we get the solution $y_1 = t(A\cos 2t + B\sin 2t)$.

For g_2, we have $m = 1$, $\mu = 1$, and μ is not a root, so the solution is $y_2 = (Ct + D)e^t$.

For g_3, it is $m = 0$ and $\mu = 0$, and since 0 is a root of multiplicity 2, we get $y_3 = Et^2$. There is a 4 in front of g_3, so we could say that it is $4Et^2$, but we might as well assume the 4 is part of E.

Therefore our solution is :

$$y(t) = At\sin 2t + Bt\cos 2t + Cte^t + De^t + Et^2$$

3. Inhomogeneous Linear Difference Equations

These are equations of the form (this example is of degree 2):

$$ax_{n+2} + bx_{n+1} + cx_n = f_n$$

The strategy for these is the same as with regular difference equations–solve the homogeneous version and then get a particular solution of the inhomogeneous equation and that gives you a general solution. You just have to plug things in and solve.

Let's solve:

$$x_{n+2} - x_{n+1} - x_n = n^2 - n - 1$$

The homogeneous equation is solved by our favorite $x_n = (1/\sqrt{5}))(\varphi^{n+1} - (1-\varphi)^{n+1})$.

To find a particular solution of the inhomogeneous equation, we plug in a polynomial of degree equal to the polynomial on the right (see p.230):

$$\begin{aligned} n^2 - n - 1 &= (a(n+2)^2 + b(n+2) + c) - (a(n+1)^2 + b(n+1) + c) - (an^2 + bn + c) \\ &= -an^2 + (2a - b)n + 3a + b - c \end{aligned}$$

Therefore $a = -1$, which forces $b = -1$, which in turn forces $c = -3$. Therefore a particular solution is $x_n = -n^2 - n - 3n$.

4. Oscillations

I highly recommend that you read whatever section each question is in, because it usually gives you helpful definitions and hints. If we're dealing with a system in which the net force applied to an object of mass m is a function $f(y, \dot{y})$ of its position (like, for instance, a spring or perhaps a buoy), then we can write

$$\begin{aligned} ma &= F \\ my'' &= f(y, \dot{y}) \end{aligned}$$

As long as f is linear in $y, \dot{y}$ (which it usually is), then this gives us a second order linear differential equation for y, which we can solve in the usual way by finding roots of the associated quadratic. It's also not too hard to find things such as equilibrium positions, since those will just be the positions for which the acceleration is zero, i.e. $f(y, \dot{y}) = 0$.

4.1. **Example.** Suppose we have a spring with spring constant k with mass m attached to it standing straight up. What's the equilibrium position? The equation we have is:

$$my'' = -ky - mg$$

Therefore, the acceleration is zero when $y = -mg/k$. You can also solve this to get an explicit equation. If for some reason the mass of the object changed suddenly, then we could solve the (new) equation with our initial condition being $y(0) = -mg/k$ to get the equation of motion. We could also write a new equation with a change of variables $s = y - (-mg/k)$.

4.2. **Critical Damping.** We know that the solutions to second order linear ODEs either oscillate, oscillate but decay in amplitude, or just decay. These solutions correspond to solutions of the associated quadratic equation–either purely imaginary, imaginary, or real. Note, though, that something special happens when we get a repeated real root. This corresponds to the parabola having a point of tangency on the x-axis, and it is obvious that any more damping will bring the parabola up and lead to a complex root and thus oscillation. The coefficient of $\dot{y}$ in the differential equation is the damping, and the damping such that the system has a repeated real root is the **critical damping**.

4.3. **Resonance.** If the system has a force applied to it at just the right frequency, then the oscillations will increase over time. Eventually, they can cause damage, such as when a wine glass breaks or a bridge collapses. If you are forcing a system, you can find the amplitude that you are giving it by looking at page 147.

4.3.1. *Example.* Suppose we have a system that satisfies $y''+y'+(5/4)y = 0$. The quadratic is $k^2+k+(5/4) = 0$, giving us roots $-(1/2) \pm i$, so the solution is $y(t) = e^{-(1/2)t}(A\cos(t) + B\sin(t))$. The natural frequency is then 2π. If we force the system with the same frequency, ie $y'' + y' + (5/4)y = cos(t)$, then we can solve this in a few ways, but I like variation of constants: we have the two solutions $y_1 = e^{-(1/2)t}\cos(t)$ and $y_2 = e^{-(1/2)t}\sin(t)$. The Wronskian is e^{-t} (after simplifying), so

$$u_1(t) = -\int e^t e^{-(1/2)t}\sin(t)cos(t)dt = (1/17)e^{t/2}(\sin(2t) - 4\cos(2t))$$

$$u_2(t) = \int e^t e^{-(1/2)t}\cos(t)cos(t)dt = (1/17)e^{t/2}(17 + \cos(2t) + 4\sin(2t))$$

So the point is that the solution $y(x)$ will **not** decay. If we didn't have a damping term in there, then the solution would in fact increase in amplitude! You can check if you leave out the exponential terms, then the Wronskian is 1 and you get a $t\sin(t)$ term.

5. Linear Systems with Constant Coefficients

A linear system is a collection of simultaneous equations, like

$$\begin{cases} \dot{y} &= ax + by + f(t) \\ \dot{x} &= cx + dy + g(t) \end{cases}$$

5.1. **How to solve them.** A good way to deal with these is to combine them into a single second order equation, solve that, and go from there. Here is how to do this: we can write (see p.254 in the book)

$$y = \frac{\dot{x} - cx - g(t)}{d}$$

then we differentiate:

$$\dot{y} = \frac{\ddot{x} - c\dot{x} - \dot{g}(t)}{d}$$

And plug in to the other equation, giving us:

$$\frac{\ddot{x} - c\dot{x} - \dot{g}(t)}{d} = ax + b\frac{\dot{x} - cx - g(t)}{d} + f(t)$$

Note that this is just a second order linear differential equation for x, which we then know how to solve. After that, we can plug it in to get a first order differential equation for y.

5.2. **Going Back and Forth Between Systems of Equations and Vector Equations.** If we have a system

$$\begin{cases} \dot{x_1} &= ax_1 + bx_2 \\ \dot{x_2} &= cx_1 + dx_2 \end{cases}$$

Then we can rewrite this as

$$\dot{x} = Ax$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

We can also do this with second (and higher) order equations. For example, if we have $\ddot{x} + a\dot{x} + bx = 0$, and we set $y_1 = \dot{x}$ and $y_2 = x$, then we note that $y_2' = y_1$ and $y_1' = -ay_1 - by_2$ (using the equation), so we can write it as a vector equation $\mathbf{y} = A\mathbf{y}$, where $A = \begin{pmatrix} -a & -b \\ 1 & 0 \end{pmatrix}$.

6. Raising e to a matrix

This may seems strange at first, but it actually is a spiffy way to express the solution to a system of linear ODEs. Of course, you must be able to raise matrices to powers. We define

$$e^A = \sum_{i=0}^{\infty} \frac{A^k}{k!}$$

Which always converges, it turns out. In order to actually compute these, you need to notice patterns and stuff.

6.1. **Example.** What's e^A, where $A = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$? Note that if you write out A^n for a few n you will see that $\begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 & nt \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and therefore $e^A = \begin{pmatrix} e & 0 & te \\ 0 & e & 0 \\ 0 & 0 & e \end{pmatrix}$ (prove this by induction or by a clear example of a few n).