MATH 2A RECITATION 11/10/11

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1. Crum's problem

In two dimensions, we want to know how many polyhedra we can arrange so that each pair touches along a face. By the four-color theorem, the maximum in any such configuration is 4. What is the maximum number for 3 dimensions? In fact, it's infinite! I've provided a link on my website to the 1947 paper by Besikovitch which constructs the polyhedra.

2. Using A Computer

It's nice to be able to check your answer by having a computer draw a phase portrait for you. The only issue is that the computer probably will have a hard time deciding where the lines should go. I believe that matlab is good at drawing portraits; it draws the direction field, and then you can click to select initial conditions. Last year, I wrote a program in Sage to do this, and you can find this on last year's website. Probably the easiest way is using PPLANE (there is a link on my website).

3. Phase Portraits of Linear Systems

If you're given the linear system

$$\begin{cases} \dot{x_1} &= x_1 + 2x_2 \\ \dot{x_2} &= 2x_1 + x_2 \end{cases}$$

Then you can write it as the matrix equation $\dot{\mathbf{x}} = A\mathbf{x}$ where $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. A good way to study the behavior of this system is to draw a phase portrait, which shows you where to go once you have picked an initial condition.

If you are given an initial condition $\mathbf{x}_0$, then you can find $\dot{\mathbf{x}}$, so you could try many initial conditions and get a direction field. However, a better way to do it is to find the eigenvectors of the matrix. Note that an eigenvector will have derivative parallel to the eigenvector. You can then fill in the rest of the picture. In our example, the characteristic polynomial is $(1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3$, which has roots 3 and -1. These eigenvalues correspond to eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Therefore, we'll have derivatives pointing towards the origin parallel to the line spanned by $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and derivatives pointing away from the origin along the line spanned by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. See the picture below (please provide a picture below).

Another system you are familiar with is where is the matrix $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Then you will get purely imaginary eigenvalues! However, you know what the phase portrait looks like–it's a bunch of concentric circles. You will have to evaluate the derivative at a point to figure out which direction they go. (please provide a bunch of concentric circles)

This happens when the real part of the eigenvalues is zero. If the real part is negative, then the circles won't be circles, but rather they will spiral inward, and similarly outward if the real part is positive (please provide two spirals below)

The type of the phase portrait depends only on the eigenvalues of the matrix; a small amount of extra information (like which direction to go in a circle around the origin) may be helpful. There are a ton (7 to 11ish depending on how you count) possibilities for the types of phase portraits, and a very nice summary with pictures can be found on pages 301–302 in the textbook.

What if one of the eigenvalues is zero?! You will figure that out on your homework.

4. Linearization Near Critical Points

We don't usually study linear systems because otherwise we'd be done already. If we have a non-linear system, e.g. $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$, then we follow the instructions on p. 314:

(1) Find all the stationary points
(2) Linearise (in the US, we typically linearize, but you can do either) near the stationary points
(3) Draw the phase portrait near the stationary points
(4) Join them up

Finding stationary points means finding points at which f and g are both zero, so a particle starting at a stationary point will remain stationary. Near a stationary point, we are guaranteed by the Hartman-Grobman theorem that the phase portrait looks about the same as the linearized version (I haven't said what that is yet) *if* the eigenvalues have nonzero real part. You will show on your homework that this condition is necessary. By the way, you may find it helpful to implicitly differentiate "the equation satisfied by r" with respect to time.

Now let's see how to linearize. The idea (as usual when linearizing something) is to take the taylor expansion of f and g. We think of shifting the origin over to the stationary point, so if (x^*, y^*) is the stationary point, then we'll define $x(t) = x^* + h(t)$ and $y(t) = y^* + k(t)$, but a shift doesn't change derivatives, so $\dot{h} = \dot{f}$ and $\dot{k} = \dot{g}$. In other words, $\dot{h} = f(x^* + h)$ and $\dot{k} = g(y^* + k)$. Then we use the tayler expansion.

I found this a little confusing—just keep in mind that the linear approximation to a function f of two variables at a point (a, b) is $f(a, b) + \frac{\partial f}{\partial x}\Big|_{(a,b)}(x - a) + \frac{\partial f}{\partial y}\Big|_{(a,b)}(y - b)$, so if we do this to the f in question and write $h = x - x^*$ and $k = y - y^*$, then

$$\dot{x} = \dot{h} = f(x^*, y^*) + \frac{\partial f}{\partial x}(x^*, y^*)h + \frac{\partial f}{\partial y}(x^*, y^*)k + \cdots$$

and

$$\dot{y} = \dot{k} = g(x^*, y^*) + \frac{\partial g}{\partial x}(x^*, y^*)h + \frac{\partial g}{\partial y}(x^*, y^*)k + \cdots$$

And (x^*, y^*) is stationary, so f and g are zero there. Therefore the first terms go away, and we are left with the linear system:

$$\begin{pmatrix} \dot{h} \\ \dot{k} \end{pmatrix} = \begin{pmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{pmatrix}_{(x^*,y^*)} \begin{pmatrix} h \\ k \end{pmatrix}$$

So the behavior of the nonlinear system at the stationary point (x^*, y^*) has a phase portrait like the linearization above. We'll do an example in the next section.

5. Ecological Models

These are just nonlinear systems that we analyze using the method in the previous section. They make for nice pictures, and you can predict things like when species will become extint, etc. There are a few types of models.

5.1. **Competing Species.** These systems look like:

$$\begin{cases} \dot{x} &= x(A - ax - by) \\ \dot{y} &= y(B - cx - dy) \end{cases}$$

Notice that an increase in either species causes a negative change in the derivative of both, as if they are completing for a shared resource.

5.1.1. *Example.* I'm all for originality, but I'm also all for having eigenvalues and stationary points come out to nice numbers, so let's do the example from the book:

$$\begin{cases} \dot{x} &= x(4 - 2x - 2y) \\ \dot{y} &= y(9 - 6x - 3y) \end{cases}$$

The stationary points are $(0, 0), (2, 0), (0, 3), (1, 1)$. To solve for these points, first try $x = 0$ or $y = 0$ and get solutions for those. Then assume both are nonzero, divide by them, and get two linear equations in two variables, which you can solve with linear algebra. Next, the matrix of partials is: $\begin{pmatrix} 4 - 4x - 2y & -2x \\ -6y & 9 - 6x - 6y \end{pmatrix}$

Now let's look at each point:

- $(0, 0)$ Here the Jacobian evaluates to $\begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}$, which obviously has eigenvalues 4 and 9 along each axis (an "unstable node").
- $(2, 0)$ Here it's $\begin{pmatrix} -4 & -4 \\ 0 & -3 \end{pmatrix}$. This has eigenvalues -4 along the x-axis and -3 with eigenvector $(-4, 1)$. It is a "stable node."
- $(0, 3)$ Gives $\begin{pmatrix} -2 & 0 \\ -18 & -9 \end{pmatrix}$, which is also a stable node with eigenvalues -2 and -9 for eigenvectors $(7, -18)$ and $(0, 1)$.
- $(1, 1)$ Gives $\begin{pmatrix} -2 & -2 \\ -6 & -3 \end{pmatrix}$, with eigenvalues -6 and 1 for eigenvectors $(1, 2)$ and $(2, -3)$. It is a saddle.

Then, we take these four local pictures and connect them up. Basically, you just draw in a bunch of lines from one to the other that indicate how solution curves will behave. In this case, note that there is an unstable equilibrium at $(1, 1)$. If the populations start exactly on the curve passing through that point in the direction of the negative eigenvector, then they will reach equilibrium and survive. If they don't start along that curve, they will head towards one of the stationary points on the axes, so one of the species will become extint.

5.2. **Predator-Prey.** These are the same, except for a slight change in the equations:

$$\begin{cases} \dot{x} &= rx(k - x - ay) \\ \dot{y} &= y(-s + bx) \end{cases}$$

(Model 1, assuming the prey obeys the logistic model) or:

$$\begin{cases} \dot{x} &= kx(1 - ay) \\ \dot{y} &= y(-s + bx) \end{cases}$$

(Model 2, assuming the prey obeys the exponential model). Note that now the predators will die off without prey and that more prey has a positive effect on the predators. If they become very successful, the predators might even search the galaxy for prey species to hunt for sport. Our models don't take this, or the fact that the prey might have acidic blood and a life cycle that involves gestation inside the predator's chest, into account.

You solve these in exactly the same way as the example above, except you will get a different picture.