MATH 2A RECITATION 11/17/11

ALDEN WALKER

1. The Collatz Conjecture

Let f, a function on the natural numbers, be defined as follows:

$$f(n) = \begin{cases} n/2 & n \text{ is even} \\ 3n+1 & n \text{ is odd} \end{cases}$$

Now repeatedly apply this function to your number. This would be an example of a discrete dynamical system. For example, $6 \to 3 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$. If you try this on a few numbers, you will see that they all seem to reach 1 after some finite number of steps. The conjecture (from 1937) is that for every natural number, repeating this procedure eventually produces 1. This problem is totally open! People have, of course, done computer searches, which have verified the conjecture up to some large value, and most people think it's true, but nobody knows.

Note that there are examples of conjectures which are true for many numbers, but fail to hold for all, so sometimes such heuristic arguments fail. For example, let $\pi(n)$ be the function counting the primes less than n and $\mathrm{li}(n)$ be the logarithmic integral function $\mathrm{li}(n) = \int_0^n \frac{1}{\log x}dx$. You may know the prime number theorem, which says that these two functions are asymptotic to each other (really, it says that $\pi(n) \sim n/\ln(n)$, but it's better approximated by $\mathrm{li}(n)$, and they are all asymptotic to each other). If you compute some values, you will see that $\pi(n) < \mathrm{li}(n)$ for everything that you can compute. You might coonjecture that this is always the case. However, this isn't true! It is known, according to Wikipedia, that there is some n for which $\pi(n) > \mathrm{li}(n)$ near e^{728}. Nobody has found an actual counterexample, though.

So beware computer evidence!

2. Conservative Systems

This has definitely come up in Physics, I am sure. We model a particle in one dimensions with total energy $E = \frac{1}{2}m\dot{x}^2 + V(x)$, where $V(x)$ is the potential energy of the particle at a point x (where the potential is determined by the position because the particle lives in a potential field). The idea is that if energy is conserved, we can find the motion of the particle from the initial time onwards if we know the total energy.

Since the change in energy is zero, we know $m\dot{x}\ddot{x} = -V'(x)\dot{x}$, so $m\ddot{x} = -V'(x)$. This we can rewrite as the system:

$$\begin{cases} \dot{x} & = & y \\ m\dot{y} & = & -V'(x) \end{cases}$$

You can find the paths that particles with differing energies follow by plotting level curves of the energy function $E = \frac{1}{2}my^2 + V(x)$. However, to analyze the behavior of the particle at stationary points, you can look at the linearization of that differential system.

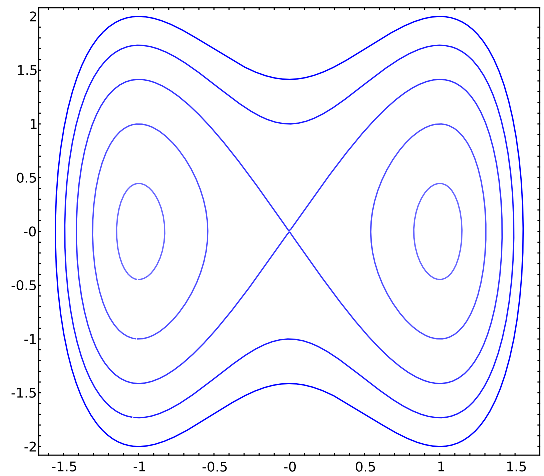
2.1. **Example.** Write down the total energy function for a particle of unit mass and a coupled system for x and $y = \dot{x}$ if the potential function is $V(x) = (x-1)^2(x+1)^2 = x^4 - 2x^2 + 1$:

Well, the total energy function is $E(x) = \frac{1}{2}\dot{x}^2 + (x-1)^2(x+1)^2$. We know that $\dot{x}\ddot{x} = -4\dot{x}x^3 + 4\dot{x}x$, so $\dot{y} = -4x^3 + 4x$.

It is not a surprise when you look at the graph of $V(x)$ that the stationary points are $(0,0)$, $(-1,0)$, and $(1,0)$. The matrix of partials is

$$D = \begin{pmatrix} 0 & 1 \\ -12x^2+4 & 0 \end{pmatrix}$$

Which, at $(0,0)$ gives a saddle with eigenvectors $(\pm 1, 2)$ with eigenvalues ± 2. At $(-1,0)$ and $(1,0)$ it's the same because of that x^2, and we have eigenvalues $\pm i\sqrt{8}$. Therefore we have centers at those two stationary points. Because the energy is constant, the system must have centers and not spirals. The phase portrait looks like:



If we are interested in finding the approximate periods of small oscillations around the points $x = \pm 1$, then we can apply the note in the class notes on page 67, which says that they have period $2\pi/\omega$, where eigenvalues are $\pm i\omega$. In our case, that would give periods of about $2\pi/\sqrt{8}$.

3. Dissipative Systems

These are the same, except we introduce a damping term into the energy function, ie now we have $m\ddot{x} = -k\dot{x} - V'(x)$, which gives us a negative derivative of energy with respect to time, and gives a system of the form (for unit mass):

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -ky - V'(x) \end{cases}$$

We study these in the same way, except now the pictures are cooler.

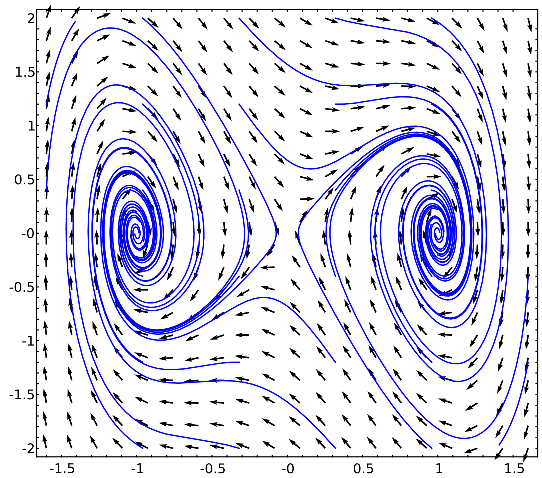
3.1. **Example.** Let's use the system from above but introduce a $-y$ term:

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -y - 4x^3 + 4x \end{cases}$$

The stationary points are the same $((0,0), (\pm 1, 0))$, and the matrix of partials is now

$$D = \begin{pmatrix} 0 & 1 \\ -12x^2 + 4 & -1 \end{pmatrix}$$

For $(0,0)$ this has characteristic polynomial $\lambda^2 + \lambda - 4$, so the eigenvalues are the rather nasty $-1/2 \pm \sqrt{17}/2$, but note this point is still a saddle, so that's good. The other points give eigenvalues of $-1/2 \pm i\sqrt{31}/2$; these points are now stable spirals. The phase portrait now looks like:



4. Lyapunov Functions and Stability of Equilibrium Solutions

If you've got yourself a dynamical system, then you can use a Lyapunov function to prove the stability of the equilibrium points. Note that this works in higher dimensions. The system $\dot{x} = v(x)$ has a **Lyapunov** function at a stationary point x^* if there is a $\mathcal{C}^1$ function (the Lyapunov function) Φ defined on a neighborhood of x^* so that:

(1) Φ has a strict minimum at x^*.
(2) $D_v\Phi(x) < 0$ for all x in some neighborhood of x^*.

Where the last condition can be replaced by $D_v\Phi(x) \leq 0$ for all x, and in that case Φ is a weak Lyapunov function. Note that the v in the directional derivative is the v from the definition of the system.
There are some facts:

- If a system has a strict Lyapunov function at x^*, then the equilibrium is asymptotically stable.
- If a system has a weak Lyapunov function at x^*, then the equilibrium is stable.
- If there is a function which satisfies condition (2) but not (1), then the equilibrium is unstable.

One way that you can think about this is to imagine the Lyapunov function standing in for the total energy (note that the total energy in the systems above satisfies the second condition). Exhibiting a Lyapunov function shows that there is some function Φ of x ($x \in \mathbb{R}^n$) such that (in a neighborhood of x^*) solutions $x(t)$ to the differential system make the quantity $\Phi(x(t))$ always decrease. If Φ has a minimum at x^*, then solutions to the system cannot escape away from the point because they can never increase Φ. Unfortunately, there is no algorithm for producing Lyapunov functions.

4.1. **Example.** Show that the critical point $(0, 0)$ of the system:

$$\begin{cases} \dot{x} &= -y - x \\ \dot{y} &= 2x^3 \end{cases}$$

is stable:
Let's try the function $\Phi(x, y) = x^4 + y^2$. Then
$D_v\Phi(x, y) = (4x^3, 2y) \cdot (-y - x, 2x^3) = -4yx^3 - 4x^4 + 4yx^3 = -4x^4$. Note that this is strictly less than 0 for all nonzero x. Also, Φ clearly has a strict minimum at $(0, 0)$. Therefore, the equilibrium at $(0, 0)$ is stable.

5. Periodic Orbits

5.1. **Dulac's Criterion.** This is a nice criterion when you want to show that there can be no periodic orbits in some region for a differential system. The criterion is: given $\dot{x} = f(x, y)$ and $\dot{y} = g(x, y)$ and a region R, then if we can find some smooth function $h(x, y)$ such that

$$\frac{\partial}{\partial x}(hf) + \frac{\partial}{\partial y}(hg) \neq 0$$

for all $(x, y) \in R$, then there are no orbits contained wholly in R. This lemma is an easy consequence of the divergence theorem, as noted in the footnotes on pp.360–361.

5.2. **Example.** Your homework problem basically shows it for all dissipative systems, but looking at the example for dissipative systems, we note that taking $h = 1$ leaves us with $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = -1 < 0$. Since this holds everywhere, we can never have a periodic orbit in that example.