MATH 2A RECITATION 12/1/11

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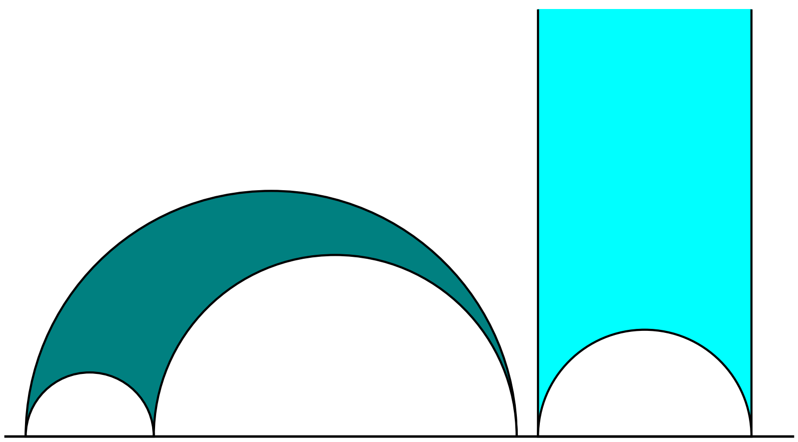
1. Hyperbolic Space

A while ago, I mentioned that if you drag vectors around on the sphere, they move around. This is a consequence of the fact that the sphere is positively curved. There are also negatively curved spaces, which are harder to visualize. By distorting things, we can draw some negatively curved spaces in the plane. In the upper half plane model for hyperbolic space, the straight lines are (1) vertical lines and (2) half-circles perpendicular to the x-axis. Here are some hyperbolic triangles:



These triangles are ideal, meaning their vertices are at infinity. The metric (distance function) in the upper half plane model is the regular Euclidean distance, scaled by $1/y^2$. Note this means that the two vertical lines are actually getting closer to each other as they get higher. Intutively, it is easier to move around at higher y values, so the fastest way to get from one point to another is to go up as high as possible.

2. Comment

As you have probably noticed, this week's homework is not in the book. You will find the notes (part 4) to be very helpful.

3. Boundary Value Problems

These are basically second order linear differential equations, just with weird requirements: instead of fixing a value and the derivative at a point, you are given some other restriction.

3.1. **Language Note.** Why are these things called eigenfunctions and eigenvalues? There is a pretty good explanation in the notes. You can think about it in this way. Let L be the linear operator on function space defined $L(u) = -u''$. Then if you have u such that $L(u) = \lambda u$, then you would say that u is an eigenfunction of the linear operator L, and the corresponding eigenvalue is λ.

This is all identical to the usual eigenvectors and values that you are used to, but because the vector space we are considering is a space of functions, we like to call the eigenvectors eigenfunctions instead.

This is why solving the differential equation $u'' + \lambda u = 0$ with boundary restrictions, and finding which values of λ are possible, is called "finding eigenfunctions" or other similar things.

3.2. **Example.** Let's do the problem: Find the eigenvalues and eigenfunctions of the BVP: $u'' + u' + \lambda u = 0$, $u(0) = 0$, $u(\pi) = 0$.

The homogeneous linear differential equation has polynomial $k^2 + k + \lambda$, which has roots $-1/2 \pm \frac{\sqrt{1-4\lambda}}{2}$. Then we have three cases:

If $1 - 4\lambda > 0$**: :** Here the solution is $u = c_1 e^{\frac{-1+\sqrt{1-4\lambda}}{2}t} + c_2 e^{\frac{-1-\sqrt{1-4\lambda}}{2}t}$, and the restrictions give us:

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 e^{\frac{-1+\sqrt{1-4\lambda}}{2}\pi} + c_2 e^{\frac{-1-\sqrt{1-4\lambda}}{2}\pi} &= 0 \end{aligned}$$

This means that $c_1 = c_2 = 0$, so there are no solutions (there are no eigenvalues λ with $1 - 4\lambda > 0$).

If $1 - 4\lambda = 0$**: :** Now the solution is $u = (c_1 + tc_2)e^{-t/2}$, and the constraints give

$$\begin{aligned} c_1 &= 0 \\ c_1 + \pi c_2 e^{-\pi/2} &= 0 \end{aligned}$$

So again, no nontrivial solutions

If $1 - 4\lambda < 0$**: :** Here there are actually solutions! We know $u = e^{-t/2}(c_1 \cos \frac{\sqrt{4\lambda-1}}{2}t + c_2 \sin \frac{\sqrt{4\lambda-1}}{2}t)$. Make sure to be careful about the sign of the thing inside the trig functions–it is the imaginary part, which means the quantity under the radical is the negative of the value you get from the quadratic formula.

The restrictions give:

$$\begin{aligned} c_1 &= 0 \\ e^{-\pi/2} c_1 \cos \frac{\sqrt{4\lambda - 1}}{2}\pi + e^{-\pi/2} c_2 \sin \frac{\sqrt{4\lambda - 1}}{2}\pi &= 0 \end{aligned}$$

Since $c_1 = 0$, the second equation gives us $\sin \frac{\sqrt{4\lambda-1}}{2}\pi = 0$, and therefore the values of λ which are possible is exactly the set of λ such that $\frac{\sqrt{4\lambda-1}}{2} = n$, for some integer n, that is, $\lambda = n^2 + 1/4$. We must also have $1 - 4\lambda < 0$, so $\lambda > 1/4$, so the set of possible eigenvalues is exactly $\{\lambda_n = n^2 + 1/4 \,|\, n > 0\}$. The corresponding eigenfunction (solution to the differential equation) is then $u = e^{-\pi/2} \sin n\pi t$. Note that I have omitted the constant c_2 because any multiple of an eigenvector is an eigenvector, so giving one (and the fact that the eigenspace is one-dimensional) gives you all of them.

3.3. **Other Helpful Info.** You might have a look at chapter 20 in the textbook.

4. Power Series Solutions

The idea here, and there are many applications of this, especially with Fourier series for functions, is that dealing with some infinite expansion of the function in terms of simple functions may be easier to deal with that handling the function itself.

4.1. **Example.** Find a series solution (in powers of x) to the BVP: $u'' + 2x^2 u = 0$, with $u'(0) = 0$, $u(0) = 2$.

We know that our solution is of the form $u = \sum_{i=0}^{\infty} a_i x^i$. Any solution to the problem must have $\sum_{i=0}^{\infty}(i+2)(i+1)a_{i+2}x^i + 2\sum_{i=2}^{\infty} a_{i-2}x^i = 0$. Since the coefficients must all cancel, we have that $2a_{i-2} = -(i+2)(i+1)a_{i+2}$. In addition, we know that $a_2 = a_3 = 0$

From the initial conditions, we know that $a_0 = 2$ and $a_1 = 0$. Therefore, $a_i = 0$ if $a \equiv 0(\text{mod}4)$, and if $i = 4k$, then we know $a_i = \frac{2}{-i(i-1)} a_{i-4} = \frac{2}{-i(i-1)} \frac{2}{-(i-4)(i-5)} \cdots \frac{2}{-4\cdot 3} a_0$, so we have

$$a_{4k} = \frac{(-1)^k 2^{k+1}}{\prod_{j=0}^{k-1}(i-4j)(i-4j-1)}$$

5. Fourier Series

So you know that "Fourier series" refers to the process in which you express a function (usually periodic ones are best) in terms of the basis $\{1, \cos(kx), \sin(kx)\}$. It is a fact that this is a basis for $L^2(-\pi, \pi)$, meaning functions f such that $\int_{-\pi}^{\pi}(f(x))^2 dx < \infty$, with the norm $\|f\| = \sqrt{\int_{-\pi}^{\pi}(f(x))^2 dx}$. Saying "this is a basis" for that space means that if you take an L^2 function, then its Fourier series converges *in the* L^2

norm to f. That is, $\|f - \sum_{i=0}^N a_i b_i\| = \sqrt{\int_{-\pi}^{\pi} \left(f - \sum_{i=0}^N a_i b_i\right)^2 dx} \to 0$ as $N \to \infty$. In this case, we write $f = \sum_{i=0}^\infty a_i b_i$. I'm writing b_i for the basis functions.

A critical fact that you don't need to know is that the space L^2 is not actually a vector space of functions with the norm above. This is because $\|f\| = 0$ does not imply that $f \equiv 0$. For instance, if $f(x) = 0$ for $x \neq 0$, and $f(0) = 1$, then $\int_{-\pi}^{\pi} f^2 dx = 0$. Therefore, to make sure that the "norm" above really is a norm, we redefine the space L^2 to be equivalence classes of functions, where two functions are equivalent if they differ on a set of measure zero. Essentially, we quotient the vector space by the kernel of the norm above.

You may have noticed that different books use different definitions of the norm (you will see this because then you get different formulas for the Fourier coefficients). This is all just convention—you want the vectors $\cos(kx)$ and $\sin(kx)$ to have norm 1, and you'd like to use the basis vector (function) 1. Unfortunately, $\int_{-\pi}^{\pi} \cos^2(kx)dx = \pi$ (same for sin), and $\int_{-\pi}^{\pi} 1dx = 2\pi$. Therefore, usually people either use $1/\sqrt{2\pi}$ instead of 1 and $1/\sqrt{\pi}\cos(kx)$ for the other functions, or (as we will do here), they multiply the norm by $1/\pi$, and the first coefficient is weird.

Keep in mind that all of this is the same as projecting a vector onto an orthogonal basis in the finite dimensional case!

The relevant formulas are on page 92 of the notes.

5.1. **Example.** Compute the Fourier expansion of the function $f(x) = x^2$ (made to be periodic by repeating every 2π). We know that $f = \frac{a_0}{2} + \sum_{k=1}^\infty a_k \cos(kx) + b_k \sin(kx)$, where

$$a_0 = < 1, f >= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = 2\pi^2/3$$

and

$$\begin{aligned} a_k &= < \cos(kx), f >= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx = \frac{4\cos(k\pi)}{k^2} \\ b_k &= < \sin(kx), f >= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx = 0 \end{aligned}$$

Therefore, $x^2 = \frac{\pi^2}{3} + \sum_{k=1}^\infty \frac{4\cos(k\pi)}{k^2}\cos(kx)$. Note that evaluating this at $x = 0$ gives $\frac{\pi^2}{3} = \sum_{k=1}^\infty (-1)^{k+1}\frac{4}{k^2}$.

5.2. **What Are You Supposed to do for Problem 7?** You already know how to do problem 7, but you may not know what it is asking you to do. You are supposed to: find the functions ϕ_n by solving the BVP. You are to assume that these functions are a basis for $L^2(0, 1)$ with the inner product $< f, g >= \int_0^1 fgdx$. You will probably want to scale the functions so they have norm 1, in which case you know the formula for the coefficients $a_n =< f, \phi_n > / < \phi_n, \phi_n >=< f, \phi_n >$. I think you have it from here.

Always be mindful of the fact that $\|f\| = \sqrt{< f, f >}$, not just the inner product of f with itself, so if you have functions which are not norm 1, you have be careful sometimes.

6. Applications to PDEs

Section 36 (p. 97) in the notes gives you instructions on how to do this problem. The idea is that you

(1) Assume that the solution is of the form $u = X(t)T(t)$ (i.e. it's separable)
(2) Deduce two separate BVPs for X and for T (X has constraints, T does not)
(3) Solve the BVP for X to get the eigenvalues λ_n and eigenfunctions X_n, and plug these in to solve for T for each n (i.e. T_n)
(4) Now you know your solution is $u = \sum c_n X_n T_n$, and the initial conditions on the problem tell you that $\sum c_n X_n = 100$ (a constant).
(5) Using the orthogonality of the X_n, you solve for the c_n, which gives you a solution.

Just follow the notes. You should write out the argument though!

6.1. **Example.** Let's do the string vibration example with the string constant $c = 1$ for a string on $[0, \pi]$ which we pluck:

$$u_{xx} = u_{tt}$$

Where $u(0,t) = u(\pi,t) = 0$, $u(x,0) = \sin(x)$ and $u_t(x,0) = 0$ (we pull the string out to the graph of sin and let it go, and it's fixed at the ends). Let's assume that $u(x,t) = X(x)T(t)$. We'll find lots of solutions of this form which satisfy the boundary conditions and then hope that a linear combination satisfies the initial conditions. We know that $u_{xx} = X''(x)T(t) = u_{tt} = X(x)T''(t)$, so we can say that

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = -\lambda$$

And thus $X''(x) + \lambda X(x) = 0$ with the boundary conditions $X(0) = X(\pi) = 0$ (we can assume $T(0) = 1$). The solutions of this system depend on whether λ is positive or negative ($\lambda = 0$ is trivial). If $\lambda < 0$ then the solution is an exponential, which cannot satisfy the boundary conditions. If $\lambda > 0$, then the solution is $\sin(\sqrt{\lambda}x)$, but we must have $\sqrt{\lambda}$ integral to satisfy the condition $X(\pi) = 0$. Therefore the eigenvalues are $\lambda_n = n^2$ for all $n > 0$ with eigenfunctions $X_n(x) = \sin(nx)$.

Now for each λ_n, we have the equation $T'' + \lambda_n T = 0$, which has general solution

$$T_n(t) = A_n \cos(n^2 t) + B_n \sin(n^2 t)$$

Each function $X_n T_n$ gives a solution which satisfies the boundary conditions, and any linear combination does too. We now take a sum of them and hope to find A_n and B_n which satisfy the initial conditions. That is, set

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos(n^2 t)\sin(nx) + B_n \sin(n^2 t)\sin(nx)$$

Now $u(x,0) = \sin(x)$ and $u_t(x,0) = 0$, which means

$$\sin(x) = \sum_{n=1}^{\infty} A_n \sin(nx) \quad \text{and} \quad 0 = \sum_{n=1}^{\infty} n^2 B_n \sin(nx)$$

We know from the class notes that the solutions are orthogonal, so we can find

$$A_n = \frac{(X_n, f)}{(X_n, X_n)} = \frac{\int_0^\pi \sin(x)\sin(nx)dx}{\int_0^\pi \sin^2 x dx} \quad \text{and} \quad B_n = \frac{(T_n, 0)}{n^2 (T_n, T_n)} = \frac{\int_0^\pi 0 \cdot \sin(nx)dx}{n^2 \int_0^\pi \sin^2 x dx}$$

So $A_1 = 1$ and all the other are zero, and $B_n = 0$. Thus

$$u(x,t) = \cos(t)\sin(x)$$

Which does satisfy everything. The reason it worked out so nice is that the initial conditions were so nice (0 and sin). If we have the initial velocity $u_t(x,0) = 0$, then the B_n will be zero, which is reasonable, but if we had a different "plucking function," then we'd get an infinite series for u. A more realistic function is

$$u(x,0) = f(x) = \begin{cases} x & \text{if } x < \pi/2 \\ \pi - x & \text{if } x \geq \pi/2 \end{cases}$$

Then

$$\begin{aligned} A_n &= \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)dx \\ &= \frac{2}{\pi}\int_0^{\pi/2} x\sin(nx)dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} (\pi - x)\sin(nx)dx \\ &= \frac{2}{\pi}\left.\frac{\sin(nx) - nx\cos(nx)}{n^2}\right|_0^{\pi/2} + \frac{2}{\pi}\left.\frac{\frac{nx\cos(nx) - \sin(nx)}{n} - \pi\cos(nx)}{n}\right|_{\pi/2}^{\pi} \\ &= \begin{cases} \frac{2}{\pi}\frac{\pi(-1)^{n/2}}{2n} - \frac{2}{\pi}(-1)^{n/2}(\frac{\pi}{2} - \frac{1}{n}) & \text{if } n \text{ is even} \\ \frac{2}{\pi}\frac{2(-1)^{n+1}}{n^2} & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

Don't quote me on those — the point is that the solution $u(x,t) = \sum_{n=1}^{\infty} A_n \cos(n^2 t)\sin(nx)$ is rather nasty, but at least we can solve it!