MATH 2B RECITATION 1/5/12

ALDEN WALKER

1. Info

Me: I am Alden Walker. My email is awalker@caltech.edu, my office is Sloan 156, and my office hours are Sunday 8-9pm. There are a lot (12) TAs with many office hours, so check the course website to see if there is someone with the hours you would like. Feel free to email me to ask questions or to set up another time to meet.

Website: My website is `http://www.its.caltech.edu/~awalker`. I plan to hand out notes every week, and these notes will be available on my website, in addition to other material that I think might be useful.

Feedback: You can give me anonymous feedback on my website. If I am really bad at something and you don't tell me, I will continue to be oblivious!

Homework: is due Mondays at 10am. Your lowest homework score is dropped; this is in lieu of an extension(!). To get an extension, you need a dean's note, and you need to tell your TA the night before.

Grading: Is 40% homework, 30% midterm, and 30% final.

Midterm/final: The midterm will be only on probability. The final will be only on statistics. However, stat depends on probability, so it is cumulative in that sense.

2. Counting!

It is important to know how to count things. Probability is just counting with division (not really) (but kind of).

2.1. **Ordered Sets.** How many ways are there of picking a list of three things out of a set of 10 things? There are 10 ways to pick the first, 9 to pick the second, and 8 to pick the third. Therefore, there are $10 \times 9 \times 8 = 720$ ways. Usually, we would write this $\frac{10!}{7!}$.

2.2. **Unordered Sets.** How about three things out of 10, where order doesn't matter? In this case, there are still 720 ways to pick three things, but for each triple, there are 3! ways to pick it. Therefore, there are $\frac{10!}{7!3!}$ ways to pick 3 things out of 10. This happens so often that it has a name: 10 choose 3, written $\binom{10}{3}$. In general, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

2.3. **Grouping.** This will be helpful to you in life! How many ways are there of picking 3 nonnegative numbers which sum to 10 (where order matters)? The following construction is often called "bars and stars." The right way to think about this is to think about distributing 10 amongst 3 things. You give some to summand one, some to summand two, and summand three. Take ten stars and two bars and use the bars to separate the stars into three groups, e.g. `***|*****|**`. This corresponds to $3 + 5 + 2 = 10$. Now you just have to count how many ways there are to put 2 bars into $10 + 2 = 12$ spots, or, how many ways are there of picking 2 places out of 12, where order doesn't matter. That's just $\binom{12}{2}$, as above. In general, there are $\binom{n+k-1}{k-1}$ ways of picking k nonnegative integers which sum to n, where order matters. If order doesn't matter, it's more complicated.

2.4. **More Grouping.** If you are asked (as you are on the homework) to answer a question like "what is the probability that if four people are selected at random, there are two eye colors represented," you first need to figure out all the ways that they can be selected so as to represent two eye colors. You need to count (say there are 2 colors, green and brown) all the possibilities: there could be 2 green and 2 brown, 3 green and 1 brown, or 1 green and 3 brown. Then figure out the probability of each of these (disjoint) events, and sum them.

3. Some Formulas

3.1. Inclusion-Exclusion. You saw in class that $P(A \cup B) = P(A) + P(B) - P(AB)$. That is all that you will need to solve the homework problem which is related to this (hint: write $\bigcup_{i=1}^n A_i = A_1 \cup \bigcup_{i=2}^n A_i$), but this can be generalized. You just need to count and keep a Venn diagram in your head. For example, $P(A \cup B \cup B) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$. To see this, think about each term: $P(A)+P(B)+P(C)$ gives you $P(A \cup B \cup C)$, but you double-counted $A \cap B$, $A \cap C$, and $B \cap C$, so you need to subtract $P(AB) + P(AC) + P(BC)$. However, now you have added $A \cap B \cap C$ three times and subtracted it three times, so you must add it back in once.

3.2. Multiplying. You'll see this later, but I think it is important to note (especially for the first problem) that it is NOT true in general that $P(AB) = P(A)P(B)$, so for instance, you CANNOT solve (f) by multiplying $P(A^c)P(B^c)$.

If $P(AB) = P(A)P(B)$, then the events are said to be independent.

3.3. Adding. When two events are disjoint, meaning that $P(AB) = 0$, then $P(A \cup B) = P(A) + P(B)$. This is an easy consequence of inclusion-exclusion above, but it is worth pointing out.

4. Example Problem (HW problems 1,2,4,5)

The distribution of eye color among people is approximately: 32% blue = B, 27% green = G, 41% brown = R. If three people are selected at random, what is the probability of finding all three eye colors? Of finding one OR two eye colors?

To solve the first question, let's count all the ways three people can have three different eye colors. There are $3! = 6$ ways of selecting three different colors, and each of them has a probability of $P(BGR) = P(B)P(G)P(R) = (0.32)(0.27)(0.41) = 0.0354$. I can multiply them this way because they are independent events. Since there are 6 ways of selecting three different people, and these events are disjoint, we can add the probabilities to get that the probability of picking three different eye colors is $6 \times 0.0354 = 0.213$.

There are two ways to do the second. Either we let X_i be the event of picking exactly i eye colors and note that $P(X_1) + P(X_2) + P(X_3) = 1$, so that the second question, which is asking for $P(X_1) + P(X_2)$, can be answered $P(X_1) + P(X_2) = 1 - P(X_3) = 0.7875$. Or, we calculate it directly. How many ways can we pick exactly two eye colors? In each case, two people must have one color and the other person has a different color. For example, we can pick BBG, BBR, BRB, etc. For each (ordered) pair of colors, there are three ways to pick it (like BBR, BRB, RBB). The probabilities of each of these are the same. We calculate all 6 of them:

BG: : $(0.32)^2(0.27) = 0.028$
BR: : $(0.32)^2(0.41) = 0.042$
GB: : $(0.27)^2(0.32) = 0.023$
GR: : $(0.27)^2(0.41) = 0.029$
RB: : $(0.41)^2(0.32) = 0.054$
RG: : $(0.41)^2(0.27) = 0.045$

Something that might come in handy is the Mathematica function `Permutations`, which gives all permutations of a list. So an easy way to generate that above list is to define `f[x_,y_,z_]:=(0.32)^x * (0.27)^y * (0.41)^z`, and then enumerate all permutations of $\{0, 1, 2\}$ and apply `f` to that, as in `Map[ Apply[f,#]&,Permutations[{0,1,2}]]`, or equivalently `(f@@#)&/@ Permutations[{0,1,2}]`.

Since there are 3 ways to get each of those pairs, we multiply all the probabilities by 3 and add them up, giving 0.666. This is the probability of getting exactly two eye colors.

The probability of getting exactly one eye color is the sum of the probabilities of getting each color, so it's $(0.32)^3 + (0.27)^3 + (0.41)^3 = 0.121$. Therefore, the answer to the second question is $0.666 + 0.121 = 0.787$, as we calculated before (there is rounding). Note that $0.213 + 0.666 + 0.121 = 1$.

5. Conditional Probability

The conditional probability of A given B is written $P(A|B)$, and is defined to be $\frac{P(AB)}{P(B)}$. If you think about it, this makes sense.

Remember that $P(AB)$ is not necessarily equal to $P(A)P(B)$. The actual rule is: $P(AB) = P(A|B)P(B) = P(B|A)P(A)$. It extends to: $P(A_1 \cdots A_n) = P(A_1)P(A_2|A_1) \cdots P(A_n|A_1 \cdots A_{n-1})$.

Note that independence is equivalent to $P(A|B) = P(A)$.

5.1. **Tree Diagrams.** If there are two events (like picking a bowl and then a ball out of the bowl) you can draw a tree, such as the one pictured on p.37, to figure out relevant conditional probabilities. I think this is relatively straightforward.

5.2. **Averaging.** Another nice rule is another straightforward one: $P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$

5.2.1. *Example Problem (HW Problems 3,6,7).* If I were to grade your final, you would have a probabiliy of 0.1 of failing. If John (name changed to protect the innocent) were to grade your final, you would have a probability of failing of 0.2. I grade 60% of the finals, and John grades 40%. What is your probability of failing?

Well, by averaging

$$\begin{aligned}\mathrm{P(fail)} &= \mathrm{P(fail|Alden)P(Alden)} + \mathrm{P(fail|John)P(John)} \\ &= (0.1)(0.6) + (0.2)(0.4) \\ &= 0.14\end{aligned}$$

5.2.2. *Example problem (HW Problems 3,6,7).* Prove that $\mathrm{P}(A|B) = \frac{\mathrm{P}(B|A)P\mathrm{P}(A)}{\mathrm{P}(B)}$. This is called Bayes' rule, and you will learn more about it next week. It's very powerful, but it's quite simple to prove:

$$\begin{aligned}\frac{\mathrm{P}(B|A)\mathrm{P}(A)}{\mathrm{P}(B)} &= \frac{\frac{\mathrm{P}(A\cap B)}{\mathrm{P}(A)}\mathrm{P}(A)}{\mathrm{P}(B)} \\ &= \frac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)} \\ &= \mathrm{P}(A|B)\end{aligned}$$

5.2.3. *Example problem.* Which of these pairs of events are independent? Find appropriate conditional probabilities

- The event of rolling a 5 on a die and then a 2.
- Two cards are drawn from a deck. The event that the first card is a 5 and the event that the second card is *not* a 5.
- Two cards are drawn from a deck. The event that the first card is red and the event that the second card is *not* a 5

The solutions are:

- These events are independent. The probability of rolling a 2 is the same as the probability of rolling a 2 given that you just rolled a 5.
- These events are *not independent.* Note

$$\mathrm{P(2nd\ card\ not\ 5|first\ is\ 5)} = (52-1-3)/(52-1) = 48/51 = 16/17$$

$$\mathrm{P(2nd\ card\ not\ 5|first\ not\ 5)} = (52-1-4)/(52-1) = 47/51$$

Also note that if the deck is shuffled, the probability that a card at a particular location is a 5 is 4/52. Thus, the probability that the card below the top card is not a 5 is 48/52. You can check that

$$\begin{aligned}48/52 &= \mathrm{P(2nd\ card\ not\ 5} \\ &= \mathrm{P(2nd\ card\ not\ 5|first\ is\ 5)P(first\ is\ 5)} + \mathrm{P(2nd\ card\ not\ 5|first\ not\ 5)P(first\ not\ 5)} \\ &= (48/51)(4/52) + (47/51)(48/52) \\ &= 48/52\end{aligned}$$

This tells you something about the eye color problem — the assumptions of independence that you must make are incorrect! The reason it is ok is that the populations are so large that the events of person 1 being brown-eyed and person 2 being blue-eyed are very very close to independent.

- These events *are* independent. The fact that the first card is red gives you no information about the number on it. In conditional probability, we would observe that

$$\mathrm{P}(\text{is } 5|\text{red}) = \mathrm{P}(\text{red and } 5)/\mathrm{P}(\text{red}) = (2/52)/(1/2) = 4/52 = \mathrm{P}(\text{is } 5)$$