Surface subgroups from linear programming

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Joint with Danny Calegari

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Motivation

Question (Gromov)

Does every one-ended hyperbolic group contain a surface subgroup?

The answer is “yes” for:

- Coxeter groups (Gordon-Long-Reid)
- Graphs of free groups with cyclic edge groups and $b_2 > 0$ (Calegari)
- Fundamental groups of hyperbolic 3-manifolds (Kahn-Markovic)
- Certain doubles of free groups (Kim-Wilton, Kim-Oum)

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- Fundamental groups of hyperbolic 3-manifolds (Kahn-Markovic)
- Certain doubles of free groups (Kim-Wilton, Kim-Oum)
- Random graphs of free groups:
  - HNN extensions of free group by random endomorphisms (Calegari-W)
  - Random amalgams of free groups (Calegari-Wilton)

Free groups

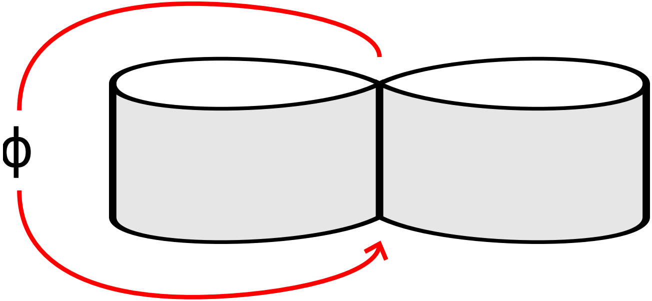
Throughout the talk, F is a free group, usually $F = \langle a, b \rangle$ of rank 2. Capital letters denote inverses: $A = a^{-1}$, $B = b^{-1}$.

(Ascending) HNN extensions

Let $F = \langle a, b \rangle$ and $\phi : F \to F$ an endomorphism.

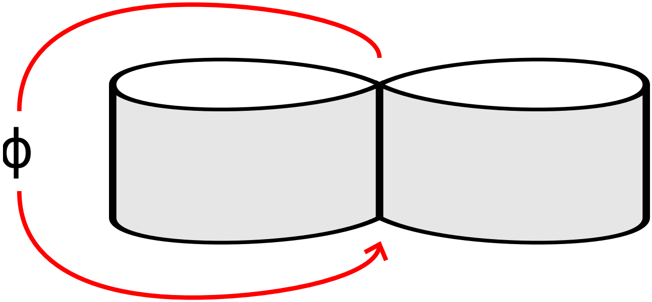
$$F_\phi = \left\langle a, b, t \,\middle|\, tat^{-1} = \phi(a),\ tbt^{-1} = \phi(b) \right\rangle$$

Topologically, it's the mapping torus of ϕ:



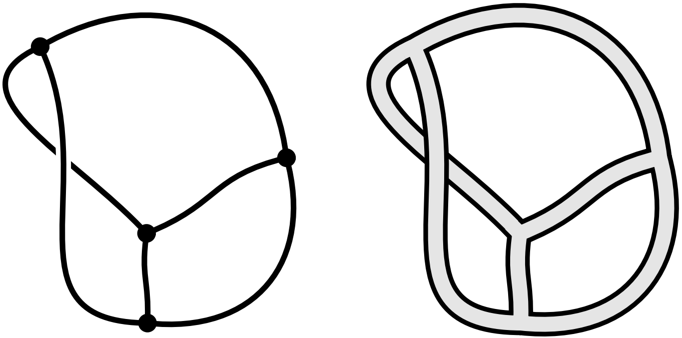
Surface maps into HNN extensions

We understand maps of closed surfaces into HNN extensions by understanding surface maps (with boundary) into free groups which "behave nicely" with respect to ϕ.



Fatgraphs

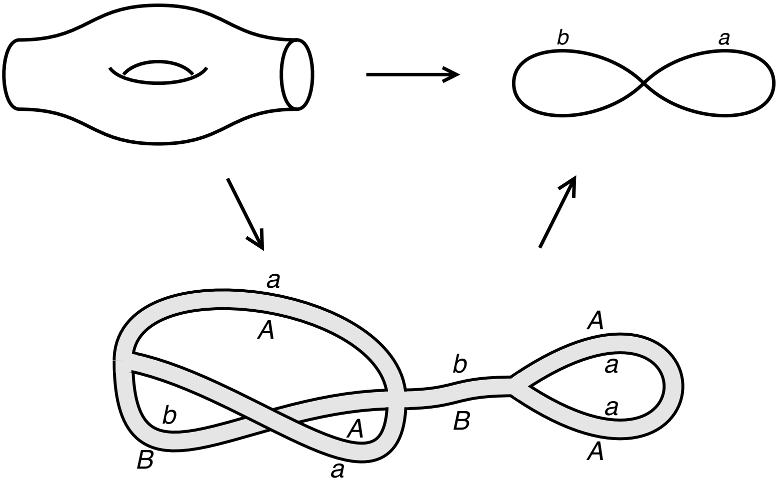
A *fatgraph* (or *ribbon graph*) is a graph with a cyclic order on the incident edges at each vertex. A fatgraph can be *fattened* to a surface.



We'll always think of our fatgraphs as already-fattened very “thin” surfaces.

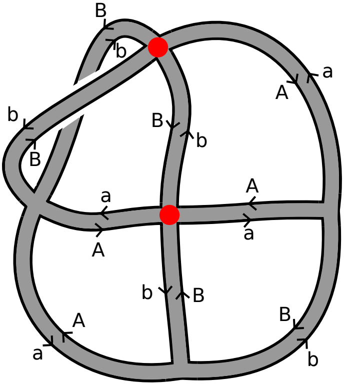
Surface maps into free groups

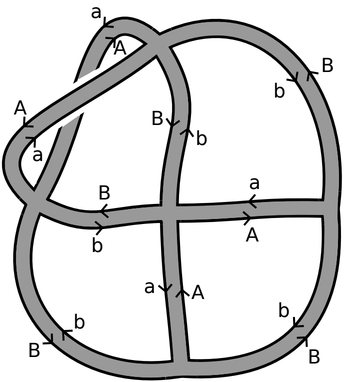
Surface maps into free groups factor through *fatgraph maps*.



Surface maps into free groups

Fatgraph maps can be (Stallings) *folded*.

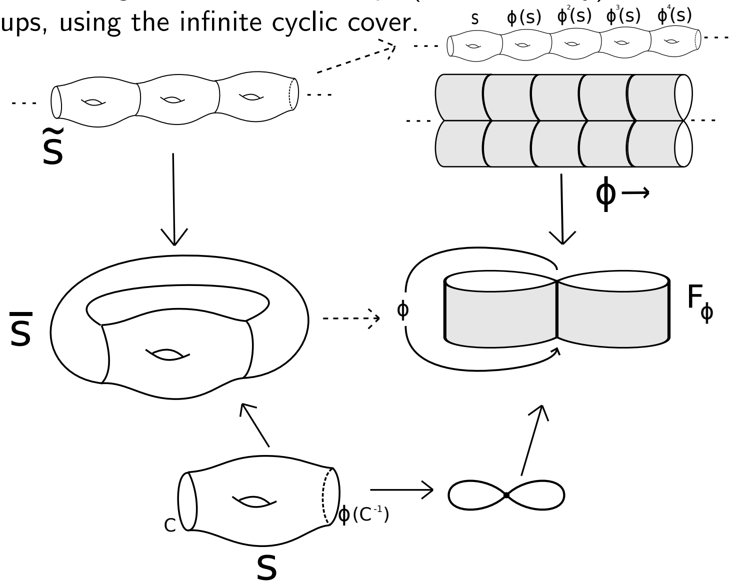




A fatgraph map which is folded is π_1-injective.

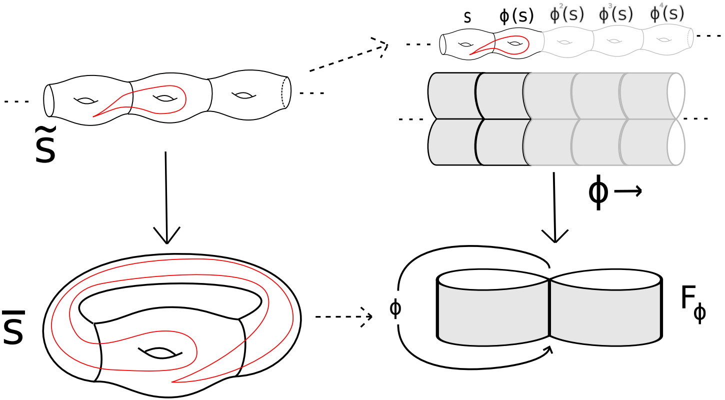
Surface maps into HNN extensions

We understand maps of closed surfaces into HNN extensions by understanding iterated surface maps (with boundary) into free groups, using the infinite cyclic cover.



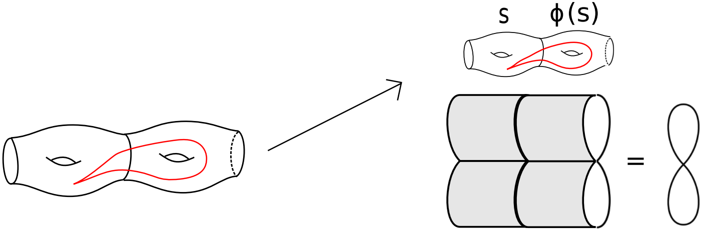
Surface maps into HNN extensions

Suppose there is a loop in $\bar{S}$ trivial in F_ϕ. Then it lifts to a compact loop in a compact part of the cyclic cover.



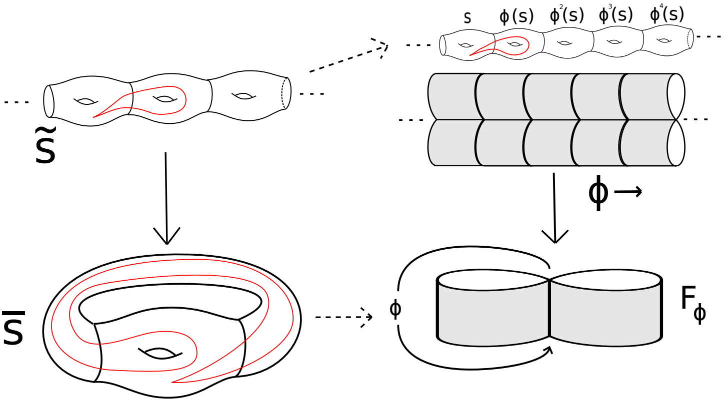
Surface maps into HNN extensions

A compact part of the cyclic cover is just the free group F, so if $\bar{S} \to F_\phi$ isn't π_1-injective, then $S \cup \phi(S) \cup \cdots \cup \phi^k(S) \to F$ isn't injective for some k.



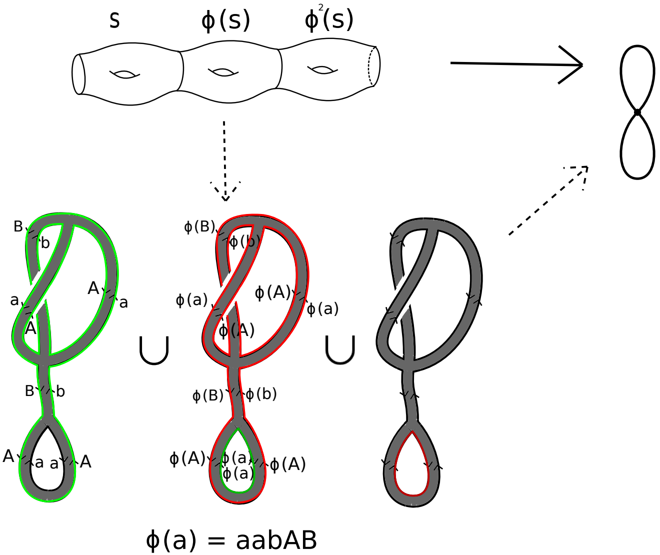
Surface maps into HNN extensions

That is, $f : \bar{S} \to F_\phi$ is injective iff all the surfaces S, $S \cup \phi(S)$, $S \cup \phi(S) \cup \phi^2(S)$, ... are injective in F.



Iterated surface maps into free groups

To check if $S \cup \phi(S) \cup \cdots \cup \phi^k(S)$ is injective in F, we can check that gluing the fatgraphs produces a *Stallings folded fatgraph*.



Iterated surface maps into free groups

Problem: gluing fatgraphs along boundaries need not even produce a fatgraph, let alone a Stallings folded fatgraph.

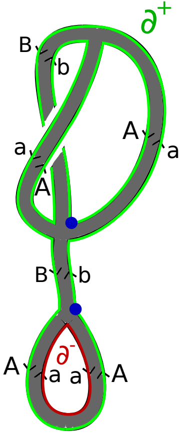
We need a combinatorial condition on the fatgraph S which ensures that gluing $S \cup \phi(S) \cup \cdots \cup \phi^k(S)$ is always a Stallings folded fatgraph.

f-folded surfaces

Consider a fatgraph Y with boundary $C + \phi(C^{-1})$. The boundary decomposes into ∂^- (loops in C) and ∂^+ (loops in $\phi(C^{-1})$).

When we glue $\phi(Y)$ to Y, we will glue $\phi(\partial^-)$ in $\phi(Y)$ to ∂^+ in Y. A vertex of ∂^+ is an f-vertex if it is in the image of a vertex in ∂^-.

In this case, the result of gluing is not folded.



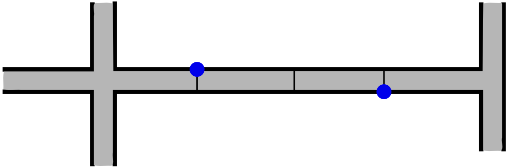
ϕ(a) = aabAB
ϕ(b) = b

● f-vertices

f-folded surfaces

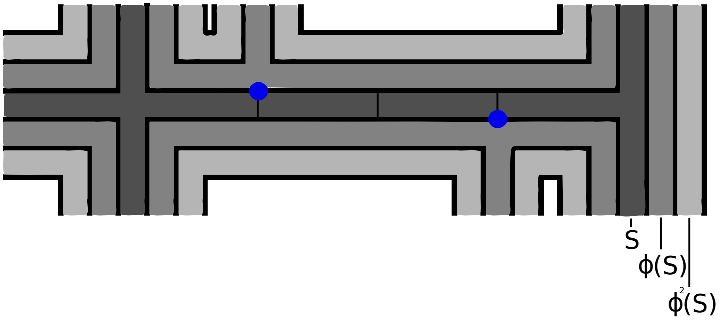
We say Y bounding $C + \phi(C^{-1})$ is f-folded if:

1. Y is Stallings folded.
2. Any vertex in Y contains at most one f-vertex of ∂^+.
3. Any vertex in Y containing an f-vertex of ∂^+ is 2-valent.
4. No vertex in Y contains more than one vertex in ∂^-



f-folded surfaces

If Y is f-folded, then $Y \cup \phi(Y) \cup \cdots \cup \phi^k(Y)$ is Stallings folded.



As the surfaces pile up, the f-folded condition ensures there is never folding and the result is a fatgraph.

f-folded surfaces

Proposition

Let Y be a fatgraph map into F with boundary $C + \phi(C^{-1})$, such that Y is f-folded. Then the map of the closed surface $\bar{Y} \to F_\phi$ is π_1-injective.

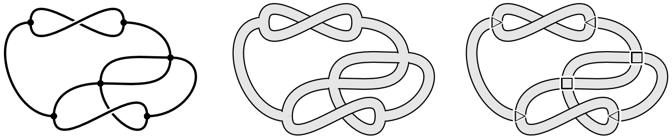
Use of the f-folded constraint

The f-folded constraint can be used *theoretically* to prove that "random" HNN extensions of free groups contain surface subgroups (the next talk).

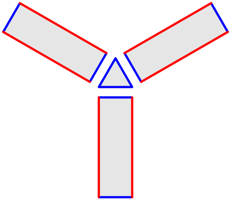
The f-folded constraint can be used *experimentally* to verify that specific HNN extensions contain surface subgroups.

Linear programming

Any fatgraph can be built out of pieces: rectangles (edges of the fatgraph) and polygons (vertices of the fatgraph).



Each rectangle has two boundary edges, and two inner edges. Each polygon has only inner edges.



Note every type of inner edge appears positively and negatively the same number of times.

Linear programming

For any given boundary $C + \phi^{-1}(C)$, there are only finitely many types of polygons and rectangles which could occur in a fatgraph with that boundary.

Consider the vector space over $\mathbb{R}$ spanned by rectangles and polygons. The condition that they can be glued up into a fatgraph is verified by checking linear equations (every inner edge appears positively and negatively the same number of times).

The f-folded constraint is local and linear, so an f-folded surface can be found by linear programming.

Example: Sapir's group

Let $\phi(a) = ab$, $\phi(b) = ba$. Then F_ϕ is Sapir's group. The surface below is f-folded, so F_ϕ contains a surface subgroup.