Surface maps into free groups

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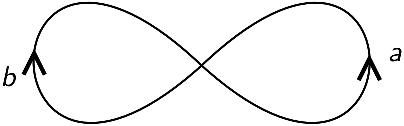
Free groups

A wedge X of two circles:



Set $F = \pi_1(X) = \langle a, b \rangle$. We write capital letters for inverse, so $A = a^{-1}$. e.g.

$$(abAABB)^{-1} = bbaaBA$$

Commutators

Let x and y be loops. The *commutator* of x and y is the loop

$$[x, y] = xyx^{-1}y^{-1}$$

so

$$[abAAB, bA] = (abAAB)(bA)(baaBA)(aB) = abAAAbaaBB$$

The set of all products of commutators is called the *commutator subgroup*, denoted $[F, F]$.

Example: the loop *abbaBAAbABBa* is in the commutator subgroup, because it is a product of commutators:

$$[ab, ba][ba, Ab] = (abbaBAAB)(baAbABBa) = abbaBAAbABBa$$

Commutators

Some random facts:

1. $[x, y]^{-1} = [y, x]$. We check:

$$[x, y][y, x] = xyx^{-1}y^{-1}yxy^{-1}x^{-1} = e$$

2. $z[x, y]z^{-1} = [zxz^{-1}, zyz^{-1}]$, since:

$$[zxz^{-1}, zyz^{-1}] = zxz^{-1}zyz^{-1}zx^{-1}z^{-1}zy^{-1}z^{-1} = zxyx^{-1}y^{-1}z^{-1}$$

Commutators

Let $|x|_z$ denote the signed number of occurrences of the letter z in the word x, so $|abAABB|_a = 1$, and $|abAABB|_B = 2$.

Lemma

For a loop $x \in F$, we have $x \in [F, F]$ iff $|x|_a = |x|_A$ and $|x|_b = |x|_B$.

So $abABAbaB \in [F, F]$.

Easy proof.

The abelianization is $H_1(F) = F/[F, F]$. A word x is trivial in the abelianzation, i.e. in $[F, F]$, iff $|x|_a = |x|_A$ and $|x|_b = |x|_B$. □

The commutator subgroup

Lemma

For a loop $x \in F$, we have $x \in [F, F]$ iff $|x|_a = |x|_A$ and $|x|_b = |x|_B$.

Direct proof.

By induction on the length of x. WLOG, assume the last letter is a. Find an A in x, and write $x = sAta$, where s and t are words. Then

$$\begin{aligned}(sAta)[(ta)^{-1}, a] &= (sAta)(At^{-1})(a)(ta)(A) \\ &= st\end{aligned}$$

So

$$sAta = st([(ta)^{-1}, a])^{-1} = st[a, (ta)^{-1}]$$

The word st is shorter than x, and still has matching numbers of a, A and b, B. By induction, st is a product of commutators, so x is.

□

The commutator subgroup

Question: if $|x|_a = |x|_A$ and $|x|_b = |x|_B$, then $x \in [F, F]$, so x is a product of commutators. What is the smallest number?

I.e. what is that smallest k so that there exist y_i and z_i so that

$$x = [y_1, z_1][y_2, z_2] \cdots [y_k, z_k]$$

We call this k the *commutator length* $\mathrm{cl}(x)$ of x.

Commutator length

Example (Culler)

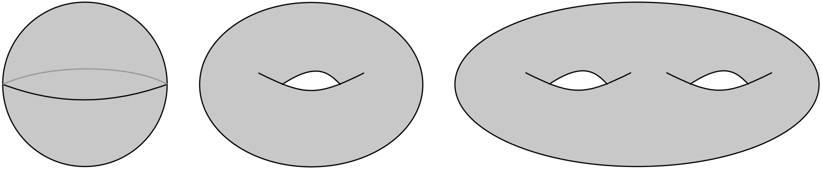
$$[x, y]^3 = [xyx^{-1}, y^{-1}xyx^{-2}][y^{-1}xy, yy]$$

So, $[x, y]^3$ can obviously be written as a product of three commutators, so $\mathrm{cl}([x, y]^3) \leq 3$. But it can secretly be written as a product of two, so $\mathrm{cl}([x, y]^3) \leq 2$.

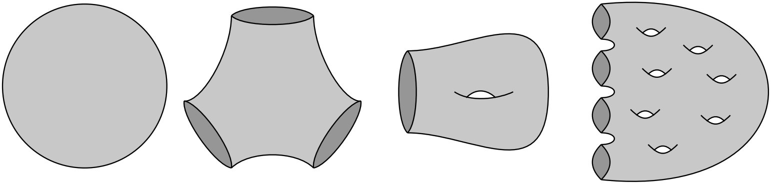
Finding $\mathrm{cl}(x)$ is a hard problem that can be solved using surfaces.

Surfaces

Some surfaces:



Some surfaces with boundary:

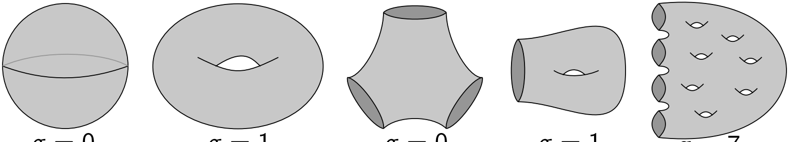


The *genus* is the number of holes.

Euler characteristic

Euler characteristic $\chi(S)$ measures the complexity of S.

$\chi(S) = 2 - 2(\text{genus}) - (\#\text{ boundaries})$



$g = 0$	$g = 1$	$g = 0$	$g = 1$	$g = 7$
$\#b = 0$	$\#b = 0$	$\#b = 3$	$\#b = 1$	$\#b = 4$
$\chi = 2$	$\chi = 0$	$\chi = -1$	$\chi = -1$	$\chi = -16$

Euler characteristic

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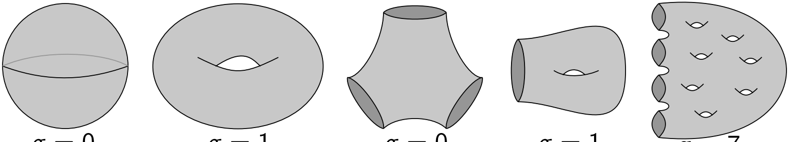

$g = 0$	$g = 1$	$g = 0$	$g = 1$	$g = 7$
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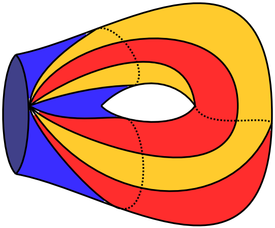
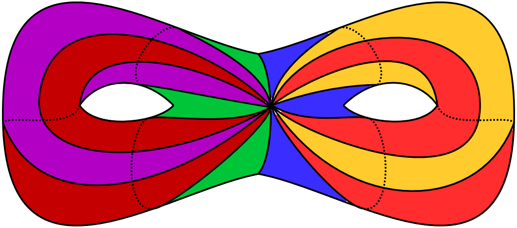
$\chi = 2$	$\chi = 0$	$\chi = -1$	$\chi = -1$	$\chi = -16$

In (any) triangulation of S, $\chi(S) = V - E + F$:

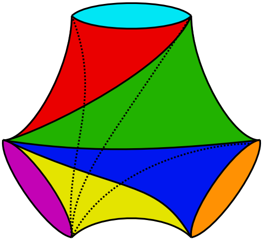


$V - E + F = 1 - 5 + 3 = -1$

More triangulations

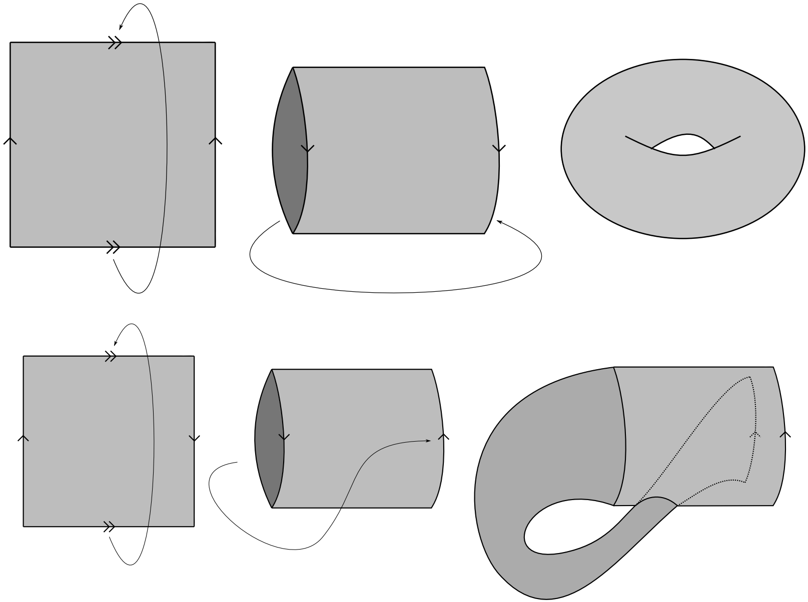


$$-2 = 2 - 2g - p = V - E + F = 1 - 9 + 6 = -2$$

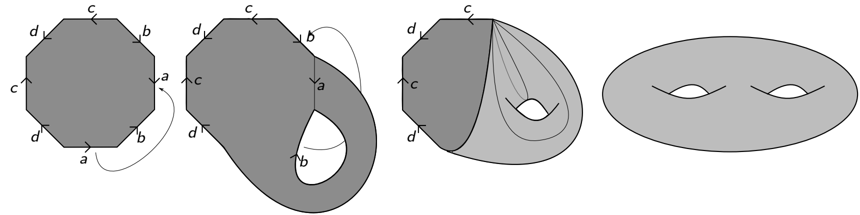


$$-1 = 2 - 2g - p = V - E + F = 6 - 15 + 8 = -1$$

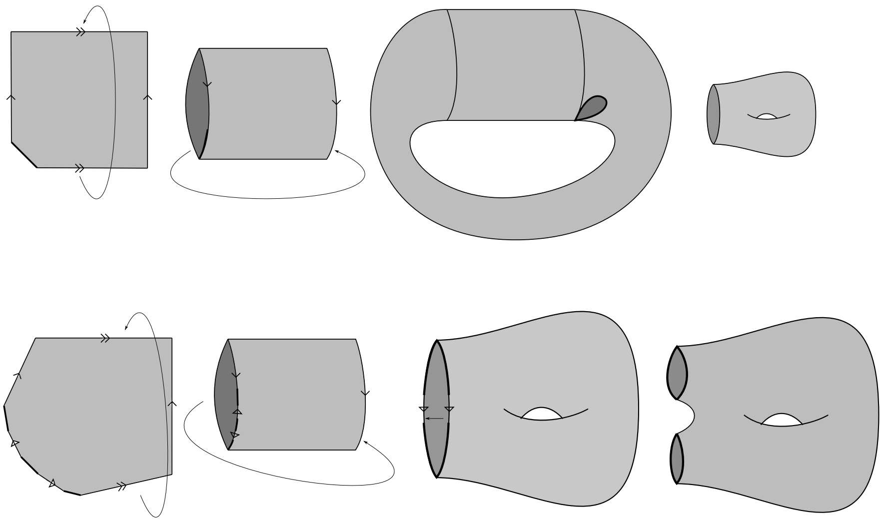
Gluing polygons to get surfaces



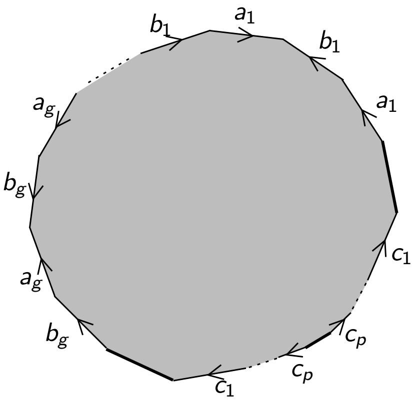
Gluing polygons to get surfaces

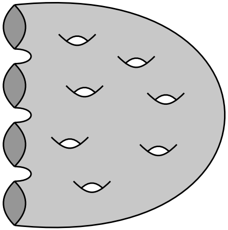


Gluing polygons to get surfaces with boundary



Gluing polygons to get surfaces with boundary

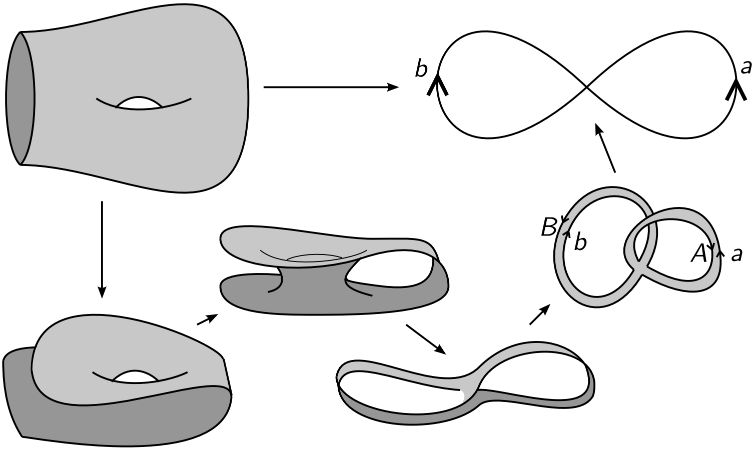




Gluing produces a genus g surface with p boundaries.

Surface maps into a free group

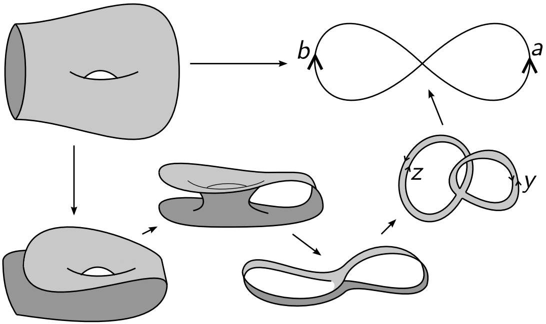
How can a surface map to a wedge of two loops?
Stretch the surface to make it skinny:



The boundary of this surface maps to the commutator $[a, b]$.

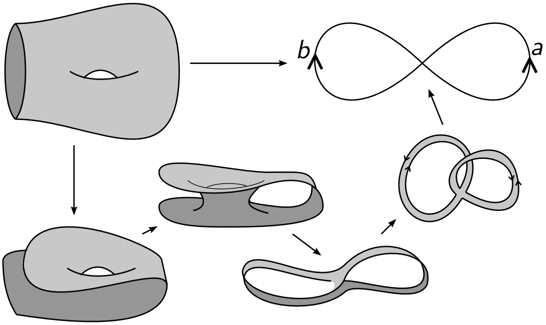
Surface maps into a free group

Suppose we have a once-punctured torus mapping into a free group so that the boundary maps to a loop $x \in F$.



Consider the two loops y and z in the surface. They map into loops y and z in F, and the boundary of the surface maps to $x = [y, z]$.

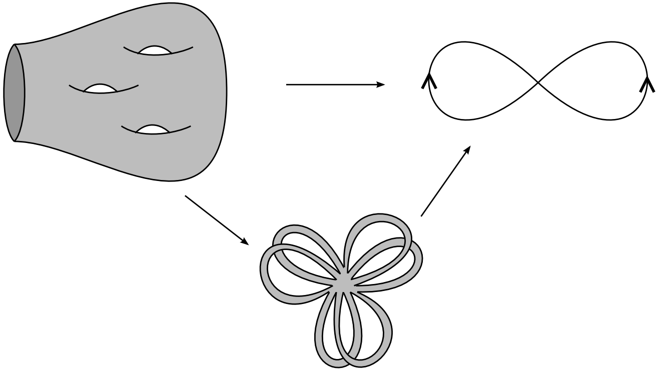
Surface maps into a free group



Lemma

For $x \in F$, x is a commutator iff there is a map of a once-punctured torus into F so that the boundary maps to x.

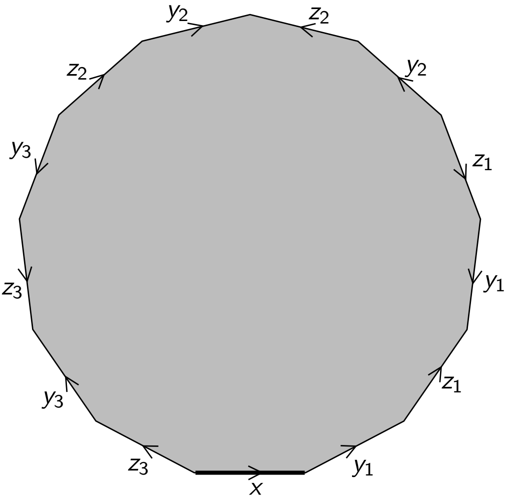
Surface maps into a free group



Lemma

In general, if S is a surface of genus g with one boundary component, then a map $S \to F$ taking the boundary of S to x is equivalent to an expression of x as a product of g commutators.

Surface maps into a free group



We can also see this by looking at a gluing polygon. Here

$$x = [y_1, z_1][y_2, z_2][y_3, z_3]$$

Finding surface maps

We are going to compute commutator length by finding surface maps.

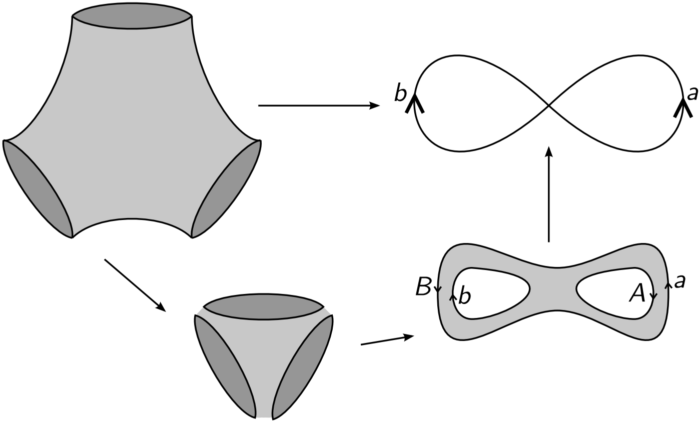
How can we find a map from a surface to a free group?

By building it out of pieces

Let us forget commutators for now and just try to find a surface map with a given boundary.

Finding surface maps

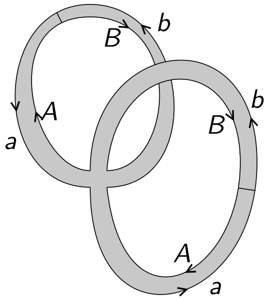
Note we could ask for a surface with multiple boundary loops.
We'll show how to build surfaces with any desired boundaries.



The skinny surface has boundary $aB + b + A$.

Finding surface maps

A *labeled fatgraph* is a graph with a cyclic order on the incident edges at each vertex. We will always draw it fattened up. A *labeled fatgraph* is a fatgraph whose edges have been labeled:



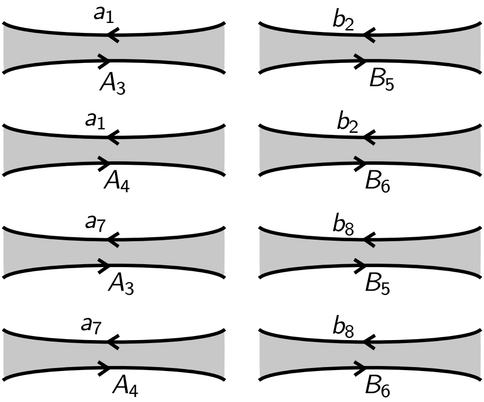
A labeled fatgraph induces a map of a surface with boundary into the free group. This map takes the boundary to *abABBAba*.

Theorem (Culler)

Every map of a surface with boundary into a free group factors through a labeled fatgraph.

Finding surface maps

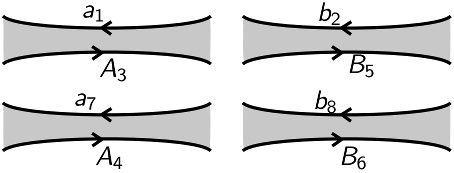
Let us look for a labeled fatgraph with boundary $abAABB + ab$. The strips (*rectangles*) that can occur are labeled with a letter-inverse pair.

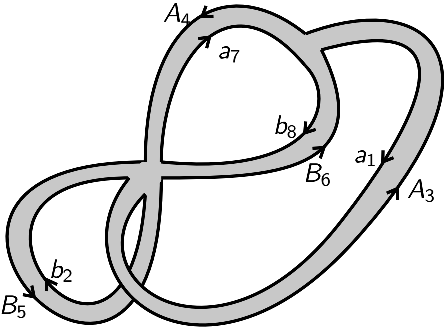


These are all possible strips; the letters are $a_1 b_2 A_3 A_4 B_5 B_6 + a_7 b_8$.

Finding surface maps

Pick rectangles that contain every letter once, and glue up:

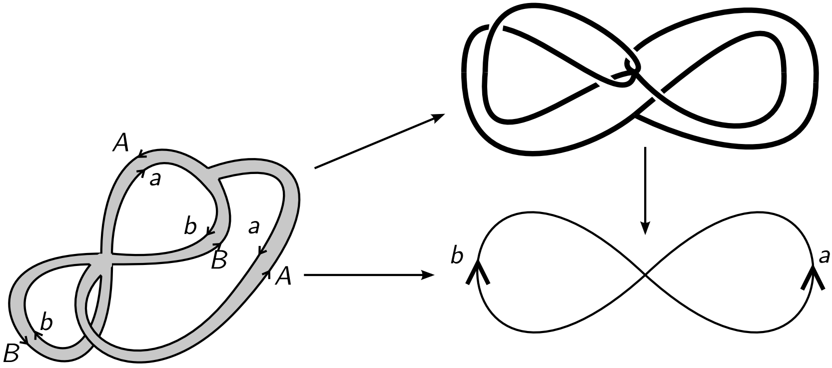




Boundary is $a_1 b_2 A_3 A_4 B_5 B_6 + a_7 b_8$.

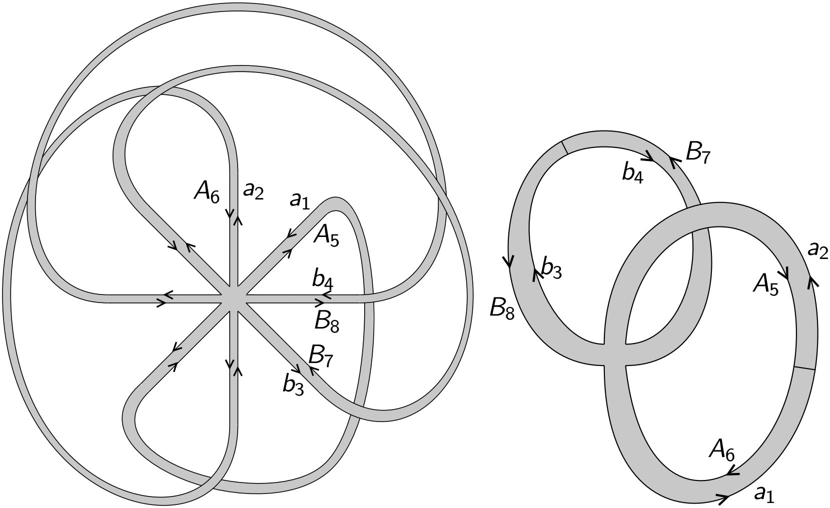
How to map labeled fatgraphs

We just built a labeled fatgraph. The labels instruct us how to get a map into the wedge of loops:



Comparing skinny surfaces

There are multiple ways to pair up the letters to get fatgraphs with a set boundary. Both of these pairings give surfaces with boundary $aabbAABB$.



Comparing fatgraphs

Recall that for a surface S, $\chi(S) = 2 - 2g - p$, and $\chi(S) = V - E + F$ for any triangulation.

Lemma

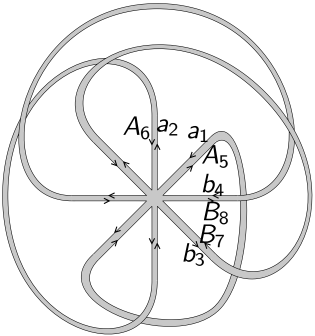
For a fatgraph S built out of J junctions and R rectangles, we have $\chi(S) = J - R$.

Proof.

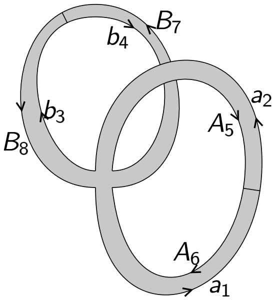
Euler characteristic is invariant under homotopy, so just homotope S to the graph with J vertices and R edges. □

Comparing fatgraphs

Therefore, we can easily compute the genus of these two surfaces

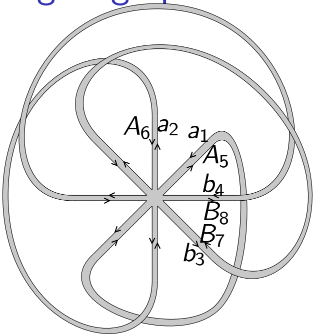


$\chi(S) = 1 - 4 = -3$
So $g = -((1-\chi)/2) = 2$



$\chi(S) = 3 - 4 = -1$
So $g = -((1-\chi)/2) = 1$

Comparing fatgraphs



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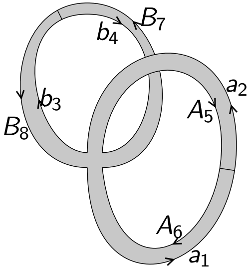


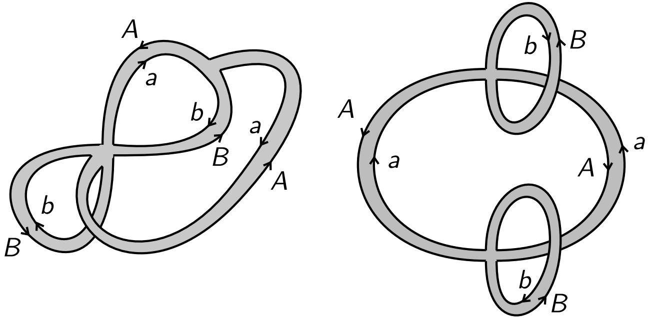
$\chi(S) = 3 - 4 = -1$
So $g = -((1 - \chi)/2) = 1$

The left surface shows that $aabbAABB$ can be written as a product of two commutators. The right shows it can be written as a single commutator. (this is obvious, since $[aa, bb] = aabbAABB$).
So $\text{cl}(aabbAABB) = 1$.

Clarification

Each surface can map into a free group in many ways. (For example, every commutator corresponds to a different map of the same once-punctured torus).

Equivalently, there are many labeled fatgraphs which are actually the same underlying surface.



These labeled fatgraphs give two distinct maps into a free group of a genus two surface with two boundaries. On the right, the boundaries are $abAABB + ab$, and on the left, $abAb + ABaB$.

Commutator length

Algorithm to compute $\mathrm{cl}(x)$:

1. Build all possible rectangles that can occur in a fatgraph with boundary x.
2. Take all possible subcollections of the rectangles so that every letter in x appears exactly once.
3. For each subcollection, glue up the rectangles, and compute the genus of the surface.
4. The smallest possible genus is $\mathrm{cl}(x)$.

Example

Recall, obviously $\mathrm{cl}([a,b]^3) \leq 3$, and being clever, we showed $\mathrm{cl}([a,b]^3) \leq 2$. Doing the algorithm proves that $\mathrm{cl}([a,b]^3) = 2$.

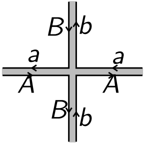
cl($[a, b]^n$)

Lemma (Culler)

$\mathrm{cl}([a, b]^n]) = \left\lfloor \frac{n}{2} \right\rfloor + 1$

Proof:

Every polygon in a fatgraph with boundary $[a, b]^n$ must have valence at least 4. This is because the order of the letters in $abAB$ means vertices simply can't close up until we see at least four rectangles.



The smallest magnitude Euler characteristic is achieved when there are as many vertices as possible, i.e. when every vertex has valence 4, so

$$-\chi(S) \geq V - V/2 = 2n - n = n$$

for a surface with boundary $[a, b]^n$.

cl([a, b]^n)

Lemma (Culler)

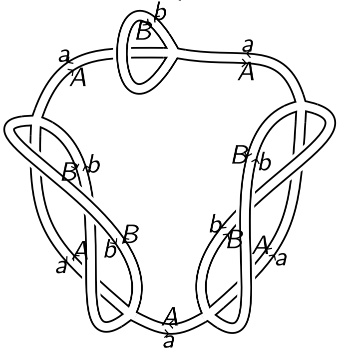
$\mathrm{cl}([a,b]^n]) = \left\lfloor \frac{n}{2} \right\rfloor + 1$

Proof continued:

Therefore, for the genus g of a surface with boundary $[a, b]^n$, we have

$$g \geq \frac{1 + (-\chi)}{2} = \frac{n+1}{2}$$

If n is odd, we can construct an explicit surface with $g = \frac{n+1}{2}$



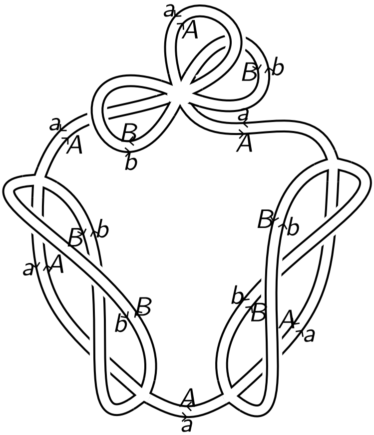
cl([a, b]^n)

Lemma (Culler)

$\mathrm{cl}([a,b]^n]) = \lfloor \frac{n}{2} \rfloor + 1$

Proof continued:

If n is even, note g must be an integer, so $g \geq \frac{n}{2} + 1$. We just add one genus to realize $g = \frac{n}{2} + 1$.



Stable commutator length

For $x \in [F, F]$, we define the *stable commutator length*

$$\mathrm{scl}(x) = \lim_{n\to\infty} \frac{\mathrm{cl}(x^n)}{n}$$

(Fact: this limit exists). We have $\mathrm{cl}(x^n) \leq n\mathrm{cl}(x)$, so $\mathrm{scl}(x) \leq \mathrm{cl}(x)$.

Example

We saw that $\mathrm{cl}([a, b]^n) = \lfloor n/2 \rfloor + 1$, so

$$\mathrm{scl}([a, b]) = \lim_{n\to\infty} \frac{\mathrm{cl}([a, b]^n)}{n} = \lim_{n\to\infty} \frac{\lfloor n/2 \rfloor + 1}{n} = \frac{1}{2}$$

Stable commutator length

Why should the limit $\mathrm{scl}(x) = \lim_{n\to\infty} \mathrm{cl}(x^n)/n$ exist?

Lemma

The sequence $\mathrm{cl}(x^n)$ *is subadditive, i.e.*

$$\mathrm{cl}(x^{n+m}) \leq \mathrm{cl}(x^n) + \mathrm{cl}(x^m)$$

Lemma (Fekete)

Let a_n *be any subadditive sequence* ($a_{n+m} \leq a_n + a_m$) *with all* a_n *positive. Then* $\lim_{n\to\infty} \frac{a_n}{n}$ *exists, and* $\lim_{n\to\infty} \frac{a_n}{n} = \inf_n \frac{a_n}{n}$.

Subadditive sequences (an aside)

Lemma (Fekete)

Let a_n be any subadditive sequence ($a_{n+m} \leq a_n + a_m$) with all a_n positive. Then $\lim_{n\to\infty} \frac{a_n}{n}$ exists, and $\lim_{n\to\infty} \frac{a_n}{n} = \inf_n \frac{a_n}{n}$.

Proof.

The sequence a_n/n is bounded below by 0, so $L = \inf_n \frac{a_n}{n}$ exists. Given $\epsilon > 0$, pick m so $\frac{a_m}{m} < L + \frac{\epsilon}{2}$ (L is the inf). Let $C = \max_{k<m} a_k$. Pick $N > m$ so that $\frac{C}{N} < \frac{\epsilon}{2}$. Now, given $n > N$, we write $n = qm + r$ for $r < m$ (quotient and remainder), and we have

$$L \leq \frac{a_n}{n} = \frac{a_{qm+r}}{qm+r} \leq \frac{qa_m + a_r}{qm+r} \leq \frac{a_m}{m} + \frac{a_r}{N} < L + \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

So $|\frac{a_n}{n} - L| < \epsilon$, as desired. □

Stable commutator length

Stable commutator length $\mathrm{scl}(x)$ measures how many commutators, on average, are required per copy of x in a large power of x. This can be smaller than $\mathrm{cl}(x)$:

Example

word x	$\mathrm{cl}(x)$	$\mathrm{scl}(x)$
$AbAbaaabAAbbABBBaaBB$	2	1
$bABBaaBAAbbabaBA$	2	3/4
$ABaBAAAbabABABaBAbaabbabAbaBaB$	2	29/24
$babaBBAbaBABABBabABaBAAbaBabbbabAAABaabA$	2	819/619
$[a, b]$	1	1/2
$[a, b]^2$	2	1
$[a, b]^3$	3	3/2
$[a, b]^4$	3	2
$[a, b]^5$	3	5/2
$[a, b]^6$	4	3

Stable commutator length

Amazing fact: it is possible to compute $\mathrm{scl}(x)$. In fact, it is *easier* to compute than $\mathrm{cl}(x)$.

Efficient surface maps

Computing scl is related to finding *efficient* surface maps.

Original question: given $x \in [F, F]$, find the surface map into F whose boundary maps to x with the smallest genus.

Better question: given $x \in [F, F]$, find the most *efficient* surface mapping to F whose boundaries all map to powers of x.

Let us think about the latter question, setting aside scl for the moment.

Efficient surface maps

What does *efficient* mean?

We say a surface S is *admissible* for x if all of the (potentially many) boundaries of S are powers of x. Given an admissible S, we let $n(S)$ be the total number of copies of x appearing in ∂S.

Compute:

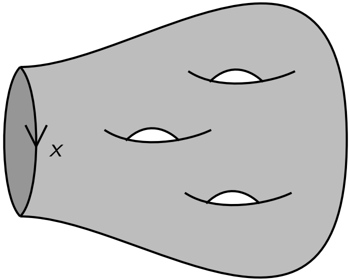
$$\inf_S \frac{-\chi(S)}{2n(S)}$$

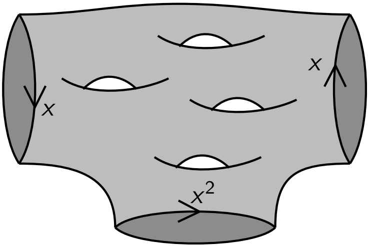
over all admissible surfaces S (there are infinitely many).

If a surface S has small $\frac{-\chi(S)}{2n(S)}$, then it has small *average complexity* per copy of x.

(It's $\chi/2n$ not χ/n because it doesn't really matter and $\chi/2n$ is approximately the genus).

Efficient surfaces





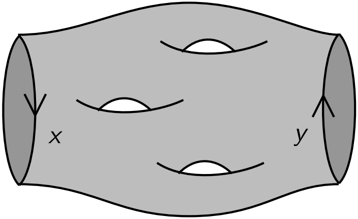
$$\chi(S) = 2 - 2(3) - 1 = -5$$
$$n(S) = 1$$
$$\frac{-\chi(S)}{2n(S)} = \frac{5}{2} = 2.5$$

$$\chi(S) = 2 - 2(4) - 3 = -9$$
$$n(S) = 4$$
$$\frac{-\chi(S)}{2n(S)} = \frac{9}{8} = 1.125$$

So while the surface on the right is more complicated, it is also more *efficient*.

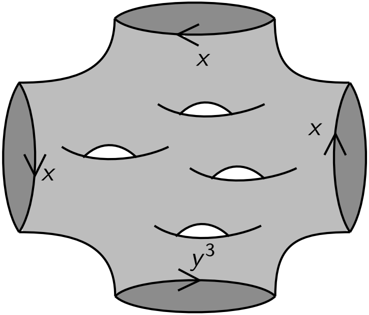
Efficient surfaces

We can search for efficient surfaces not just for a word x, but for several words x, y, z. In this case, we require that every boundary component is a power of x, y, or z, and we require that the total number of copies of each word is the same (and we denote it $n(S)$).



$\chi(S) = -6$, $n(S) = 1$

$\frac{-\chi(S)}{2n(S)} = 3$

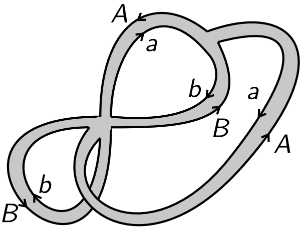


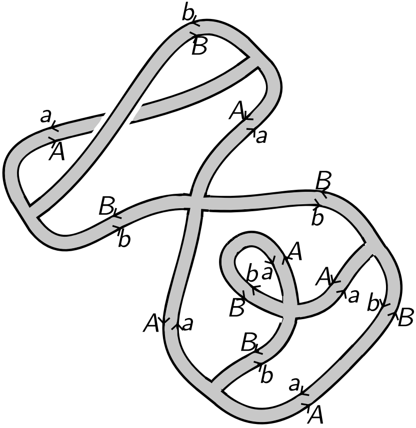
$\chi(S) = -10$, $n(S) = 3$

$\frac{-\chi(S)}{n(S)} = \frac{10}{6} < 3$

Efficient surfaces

Example: The best surface which wraps around $abAABB + ab$ once has $-\chi(S)/2 = 1$. The most *efficient* surface wraps $n = 3$ times around, and has $-\chi(S)/(2n(S)) = 2/3$.





Efficient surfaces and scl

Theorem (Calegari)

For $x \in [F, F]$,

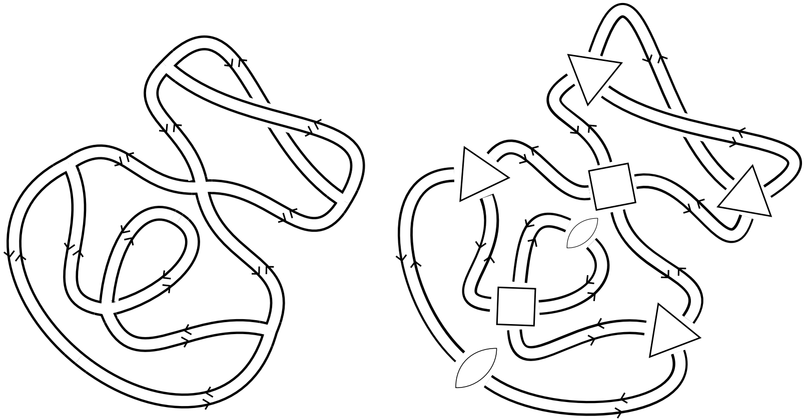
$$\mathrm{scl}(x) = \inf_S \frac{-\chi(S)}{2n(S)}$$

Over all admissible surfaces.

So if we can find efficient surfaces, we can compute scl. But there are infinitely many admissible surfaces for x; how can we compute the inf?

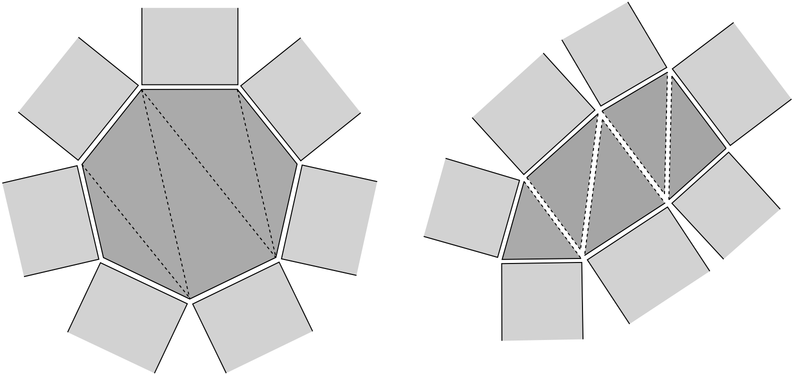
Finding efficient surfaces

Any labeled fatgraph for $abAABB + ab$ can be broken into pieces:



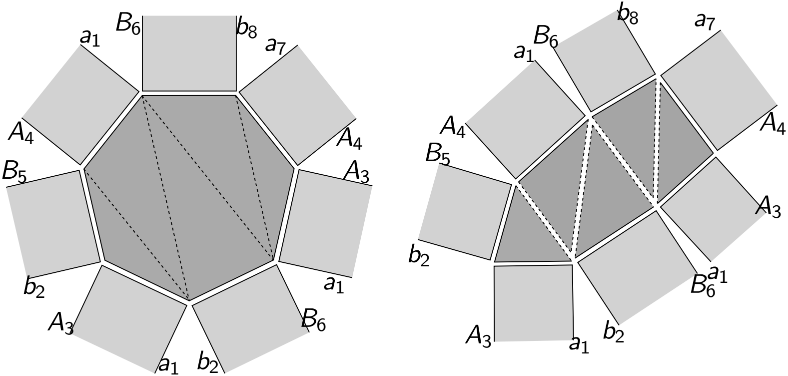
Every rectangle is one of the finitely many possibilities. But we need to understand the junctions, which we'll call *polygons*.

Polygons



Any polygon can be cut into triangles, and there are only finitely many kinds of triangles.

Polygons



B_6
b_8
a_1
a_7
A_4
A_4
B_5
A_3
b_2
a_1
A_3
B_6
a_1
b_2
b_8
B_6
a_7
a_1
A_4
A_4
B_5
A_3
b_2
a_1
B_6
A_3
a_1
b_2

Finding efficient surfaces

Hence:

1. There are finitely many kinds of rectangles and triangles which can occur in a fatgraph with boundary x.
2. Let V be the vector space over $\mathbb{Q}$ spanned by the set of rectangles and triangles for x. The conditions on a vector which ensure that we can glue it up are *linear*.
3. If $v \in V$ is a vector recording how many rectangles and triangles we have, and it can be glued up, then the Euler characteristic of the resulting fatgraph is independent of how we glue, and it is linear in v.

These facts mean that

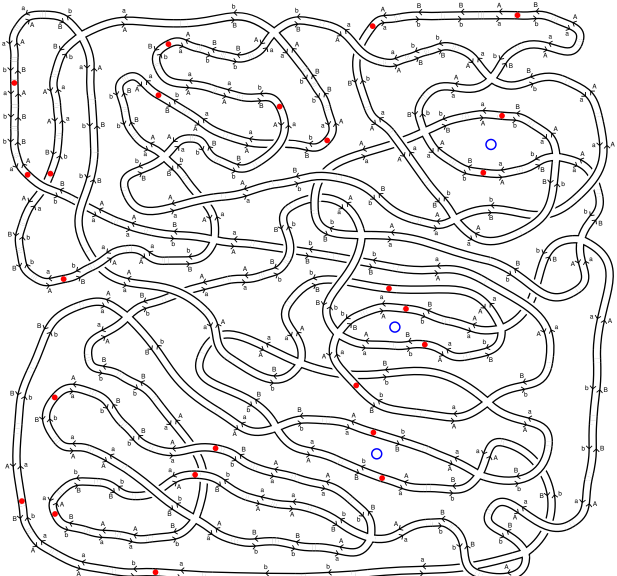
Theorem (Calegari, W)

For $x \in [F, F]$, the value of $\mathrm{scl}(x)$ is the solution to a linear programming problem whose size is polynomial in the length of x.

Furthermore, a most-efficient surface exists (the inf is realized).

Linear programming example

Here is a big fatgraph, giving a map of a surface with 4 boundery components into a free group:

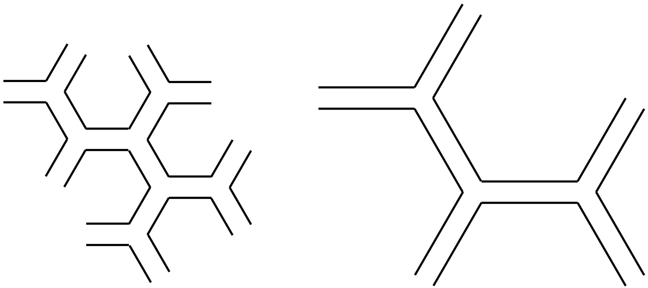


Visualizing efficient surfaces

Suppose we are given a word x. For any fatgraph S admissible for x, we have $-\chi(S) = (\#\text{ edges}) - (\#\text{ vertices})$, and we want to minimize

$$\frac{-\chi(S)}{2n} = \frac{(\#\text{ edges}) - (\#\text{ vertices})}{2n}$$

So for a given boundary length, a surface with *long* edges is efficient (because there will be *fewer* edges):



Example theorems

Let x be a long random word in $[F,F]$ with length n.

Theorem (Calegari-W)

Let F have rank k. For any $\ell < 1$, there is a $C > 1$, so that with probability $1 - O(n^{-C})$, there is a trivalent fatgraph with boundary x whose average edge length is

$$\ell \frac{\log(n)}{2\log(2k-1)}$$

Further, for any $\ell > 1$, there is a $C > 1$ so that with probability $1 - O(n^{-C})$, there is no trivalent fatgraph with the given average edge length.

Note this implies $\mathrm{scl}(x) \approx \frac{n}{\log(n)} \frac{\log(2k-1)}{6\ell}$.

Theorem (Calegari-W)

For any L, there is a $c > 0$ such that with probability $1 - O(e^{-n^c})$ there exists a trivalent fatgraph with boundary x and with every edge of length at least L.