Homologically essential surface subgroups of random groups

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Definition (Gromov's density model for random groups)

Fix a rank k and a *density* $0 \leq d \leq 1$. Let F_k be a free group of rank k. Define

$$G_n = \langle F_k \mid R \rangle$$

Where the set of relators R is $(2k-1)^{dn}$ words chosen uniformly at random from all words of length n.
We say that a *random group has property* P if

$$\Pr(G_n \text{ has } P) \to 1 \text{ as } n \to \infty.$$

Theorem (Calegari-W)

A random group at density $0 \leq d < 1/2$ *contains a quasiconvex surface subgroup. If* $d > 0$*, then this subgroup can be taken to be the image of a homologically essential map of an orientable surface.*

Theorem (Calegari-Wilton)

A random group at density $d < 1/2$ *contains a subgroup isomorphic to a 3-manifold with totally geodesic boundary.*

Theorem (Calegari)

A random group at density $d < 1/2$ *contains quasi convex subgroups commensurable with infinite families of Coxeter groups.*

Question (Gromov)

Does every one-ended hyperbolic group contain a surface subgroup?

The answer is "yes" for:

- Coxeter groups (Gordon-Long-Reid)
- Graphs of free groups with cyclic edge groups and $b_2 > 0$ (Calegari)
- Fundamental groups of closed hyperbolic 3-manifolds (Kahn-Markovic)
- Certain doubles of free groups (Kim-Wilton, Kim-Oum)
- Random graphs of free groups:
  - HNN extensions of free group by random endomorphisms (Calegari-W)
  - Random amalgams of free groups (Calegari-Wilton)

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- Random groups at density $0 \leq d < 1/2$ (Calegari-W)

Comparison with Kahn-Markovic surfaces

The original Kahn-Markovic surfaces in hyperbolic 3 manifolds are homologically trivial. They take two copies of each pair of pants with opposite orientations in order to make it easier to glue them up. It is harder in the case of

Theorem (Kahn-Markovic)

The Ehrenpreis conjecture.

Theorem (Liu-Markovic)

Every second homology class in a closed hyperbolic 3*-manifold is represented by a* π_1*-injective surface map. Every homologically trivial* 1*-manifold bounds a* π_1*-injective surface.*

Comparison with Kahn-Markovic surfaces

Similarly:

Theorem (Calegari-W)

A random group at density $0 \leq d < 1/2$ contains a surface subgroup.

Uses a trick; harder work gives:

Theorem (Calegari-W)

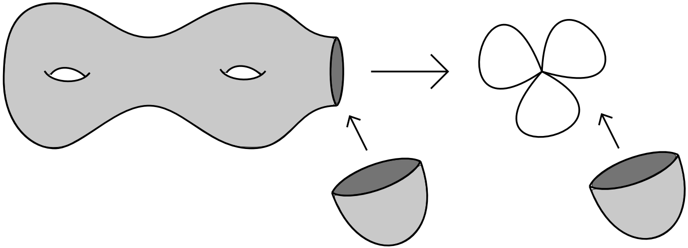
A random group at density $0 < d < 1/2$ contains a homologically essential surface subgroup.

There are analogies between (a small part of!) these proofs and Kahn-Markovic, although these analogies don't appear to lead to anything new.

Theorem (Calegari-W)

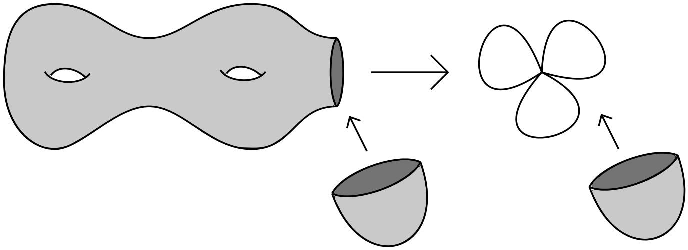
A random group at density $d < 1/2$ contains a quasiconvex surface subgroup. If $d > 0$, then this subgroup can be taken to be the image of a homologically essential map of an orientable surface.

General proof strategy: Build a map f of a surface S with boundary into F_k such that $f(\partial S)$ is the relator r. In G_n, then, the map extends over a disk, giving a map of a closed surface S' into G_n.



If r is not homologically trivial in F_k, we'll build a surface with boundary $r + r^{-1}$. If r is homologically trivial in F_k, then the map $f : S' \to G_n$ is homologically essential.

Main question for the proof:

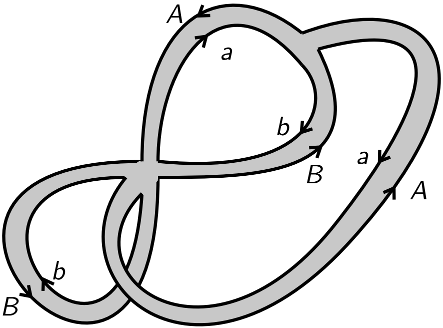


- How can we build a map $f : S \to F_k$ so that $f : S \to F_k/\langle\langle R \rangle\rangle = G_n$ is π_1-injective?

There are many steps, but a key result is the *thin fatgraph theorem*.

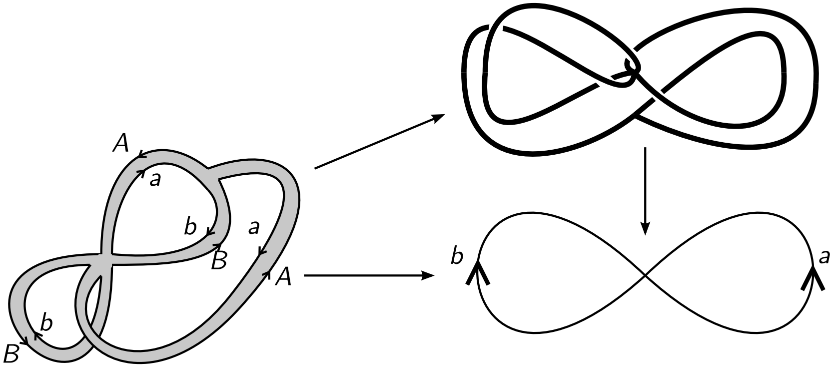
Fatgraphs

A *fatgraph over* F_k is a graph with a cyclic order on the incoming edges and edges labeled by generators of F_k (here a, b).



A fatgraph can be fattened into a surface whose boundary is decorated with words in F_k.

Fatgraphs



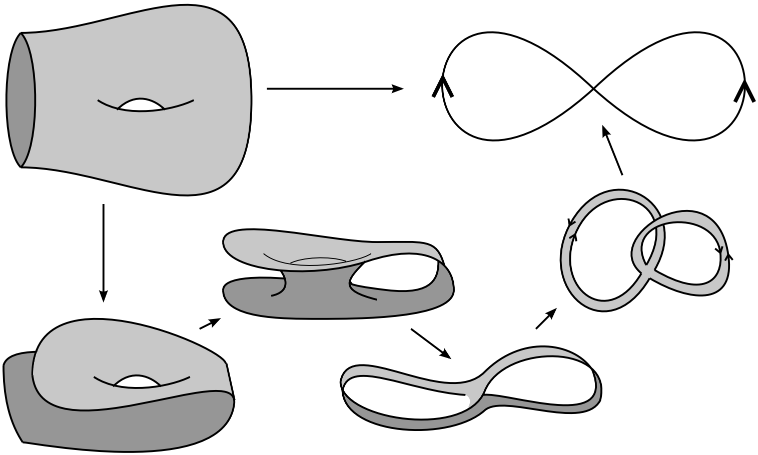
The labeling on a fatgraph over F_k induces a map of the surface with boundary into F_k.

Lemma (Culler)

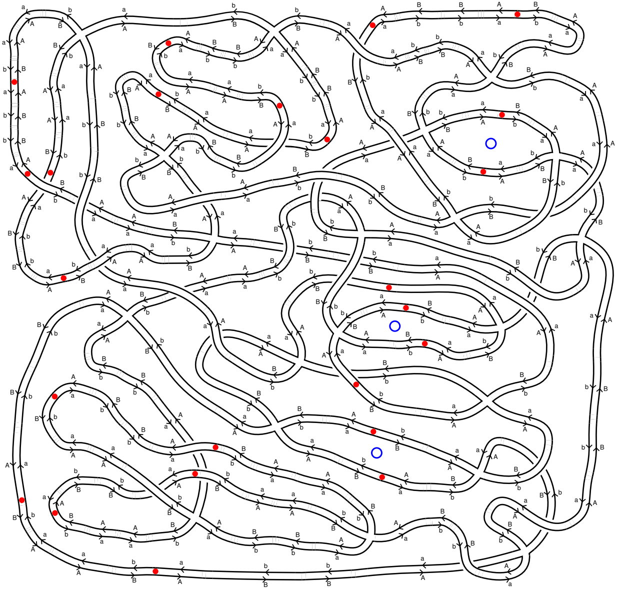
After compression and homotopy, every surface map into a free group is a fatgraph map.

Proof.

Make the surface skinny.



Fatgraphs can be quite big



Notice how this fatgraph has many long edges (sequences of 2-valent vertices).

The thin fatgraph theorem

Theorem (Calegari-W)

For any $L > 0$, there is $C > 0$ so that given a random word w of length n, there is a trivalent fatgraph with boundary w and with all edges of length at least L with probability $1 - O(e^{-Cn})$.

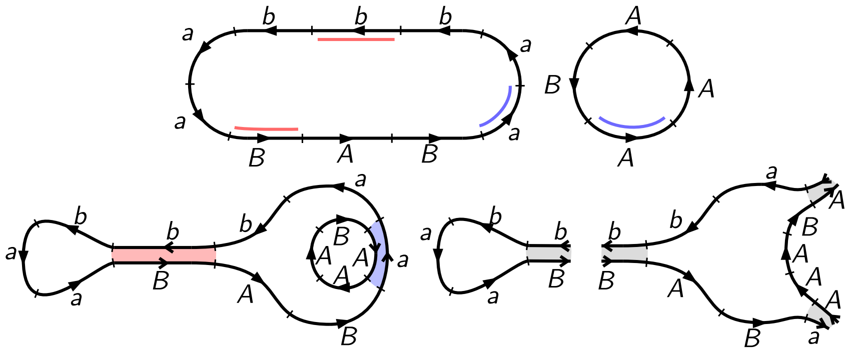
I.e. A long random word is the boundary of a *sparse* fatgraph.

This theorem is the L^∞ version of the theorem:

Theorem (Calegari-W)

In a free group of rank k, there is $C > 0$ so that with probability $1 - O(n^{-C})$, a random word of length n is the boundary of a fatgraph with average edge length $\log(n)/2\log(2k-1)$.

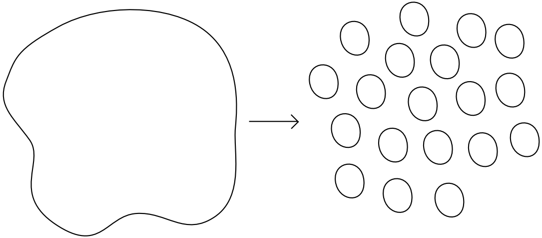
The thin fatgraph theorem proof



To build a fatgraph with desired boundary, we can proceed by gluing small portions of the loops, one at a time. After gluing a small amount of our loops, we obtain a partial fatgraph and the remainder loops. Then we glue portions of the remainder, etc.

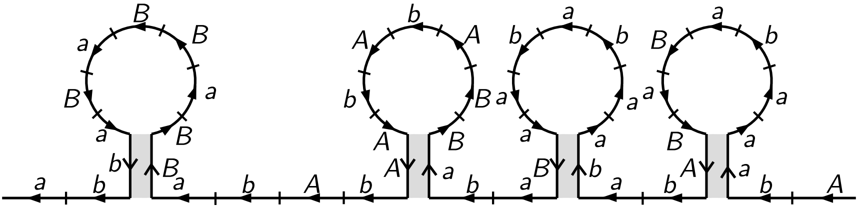
The thin fatgraph theorem proof

We want to glue up a long random word. The trick is to use a sequence of gluings to turn a single long word into a huge number of *almost equidistributed* remainder loops which are *uniformly bounded* in size.

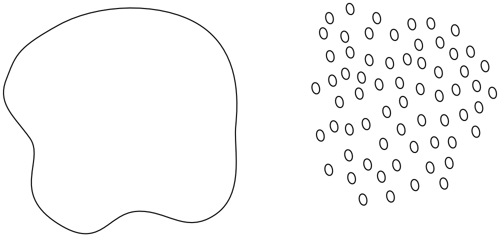


Trading a big loop for little loops

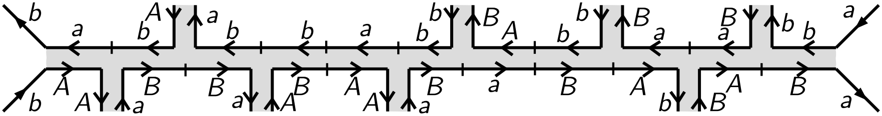
Step 1: Pinch off short loops:



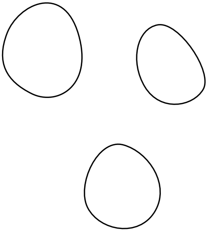
The remainder is now one big loop and a “reservoir” of equidistributed short loops:

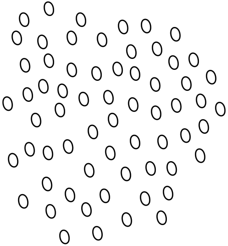


Step 2: Glue aligned long segments:

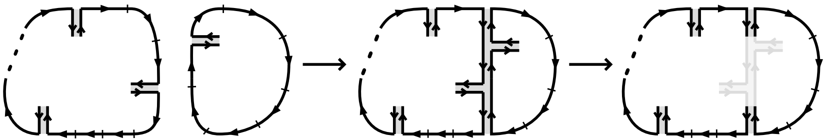


The remainder is now several loops, where we can't control the length of each loop, but the *total length* is small compared to the size of the reservoir.

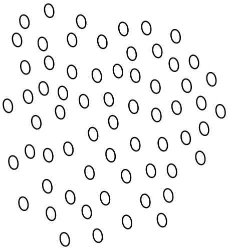




Step 3: Gluing the remainder:
The total length of the reservoir is large compared to the remainder, so we can assemble exactly the inverses of our remainder loops.



Now there is only the reservoir, a large, almost equidistributed collection of loops of a fixed size.



Then either

- Take two copies of the reservoir, pair them up to get a nonorientable surface, take a double cover to get a homologically trivial π_1-injective surface.
- Apply the theorem:

Theorem (Calegari-W)

For any L, a sufficiently equidistributed collection of loops of length $4L$ is the boundary of a fatgraph with edges of length L.

To get a homologically essential π_1-injective surface.

Hence,

Theorem (Calegari-W)

A random group at density $d < 1/2$ contains a quasiconvex surface subgroup. If $d > 0$, then this subgroup can be taken to be the image of a homologically essential map of an orientable surface.