Further exploring the parameter space of an IFS

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Joint with Danny Calegari and Sarah Koch

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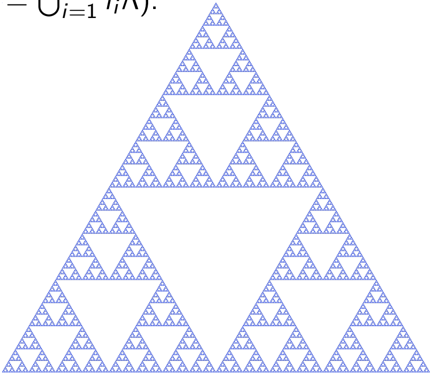
Recall: Sarah Koch spoke about this project on September 26. I will give essentially the same introduction, but I'll discuss some different topics in the second half.

Goal: Given dynamical systems parameterized by $c \in \mathbb{C}$, connect features of the dynamics to features of the parameter space $\mathbb{C}$ and to other areas of math.

Our dynamical systems of interest are *iterated function systems*:

Iterated function systems (IFS):

Let $f_1, \cdots, f_n : X \to X$, where X is a complete metric space, and all f_i are contractions. Then $\{f_1, \ldots, f_n\}$ is an *iterated function system*. We are interested in the *semi*group generated by the f_i. There is a unique invariant compact set Λ, the *limit set* of the IFS. ("Invariant" here means that $\Lambda = \bigcup_{i=1}^{n} f_i \Lambda$).



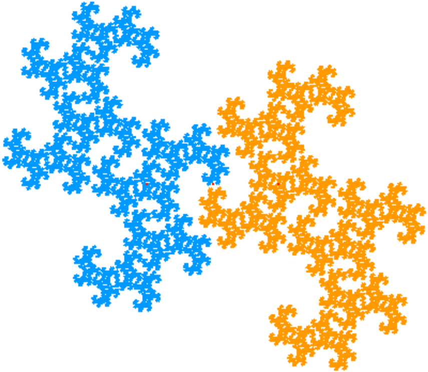
(pictures from Wikipedia)

Parameterized IFS

For $c \in \mathbb{C}$ with $|c| < 1$, consider the IFS generated by the dilations

- $f_c(z) = cz - 1$; (centered at $\alpha_f = -1/(1-c)$)
- $g_c(z) = cz + 1$; (centered at $\alpha_g = 1/(1-c)$)

The limit set Λ_c will have symmetry around $(\alpha_f + \alpha_g)/2 = 0$



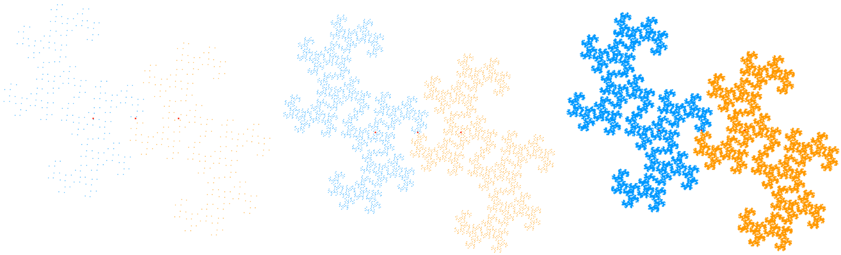
Note $\Lambda_c = f_c\Lambda_c \cup g_c\Lambda_c$; we draw these sets in blue and orange.

How to compute Λ_c

There are two simple ways to construct Λ_c. Let G_n be all words of length n in f_c, g_c.

Method 1: Let p be any point in Λ_c (for example the fixed point of f_c, i.e. $-1/(1-c)$). Then

$$\Lambda_c = \overline{\bigcup_n G_n p}$$



How to compute Λ_c

Method 2: Let D be a disk at 0 which is sent inside itself under f_c and g_c. Let

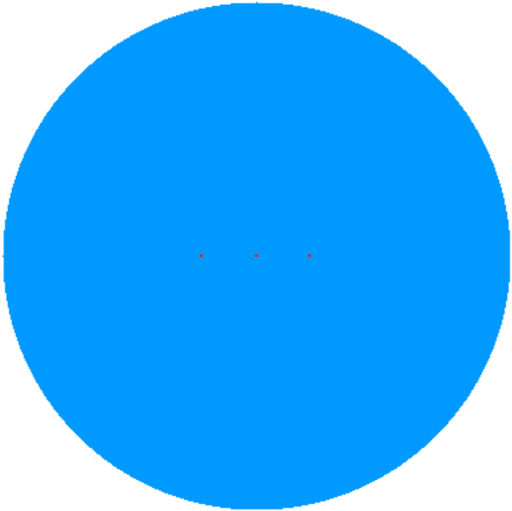
$$G_n D = \bigcup_{u \in G_n} uD$$

Then for *any* n, we have $\Lambda_c \subseteq G_n D$, and

$$\Lambda_c = \bigcap_{n \geq 0} G_n D$$

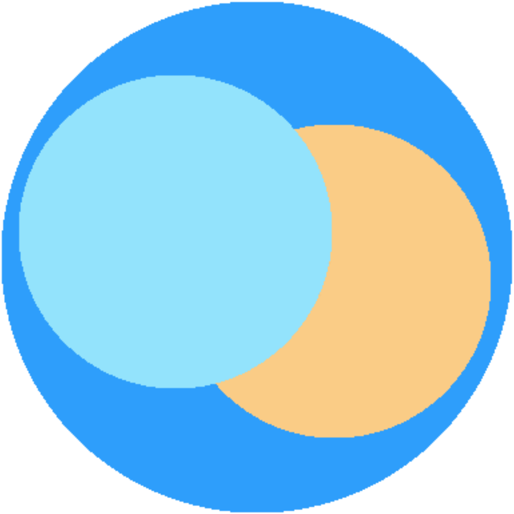
Λ_c

D:



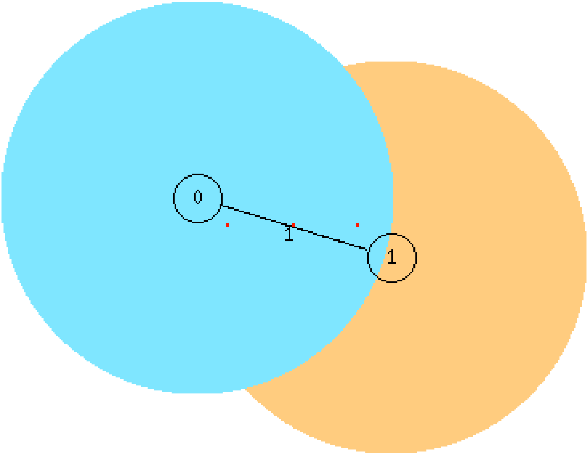
Λ_c

D is sent inside itself under f_c and g_c:



Λ_c

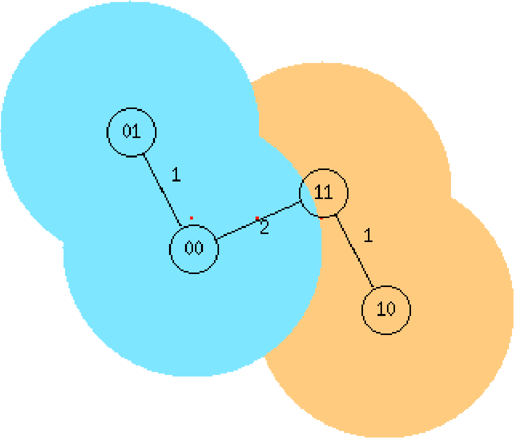
$G_1 D$, i.e. $f_c D \cup g_c D$:



Here $0 = f_c$, $1 = g_c$

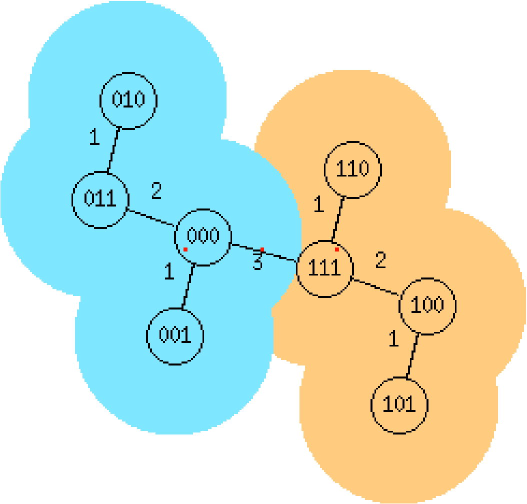
Λ_c

$G_2 D$, i.e. $f_c f_c D \cup f_c g_c D \cup g_c f_c D \cup g_c g_c D$:



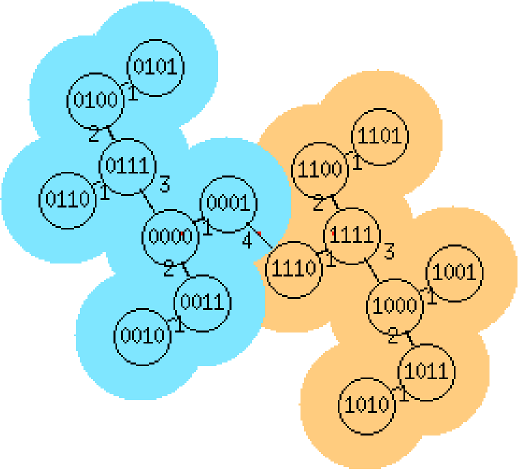
Λ_c

G_3D:



Λ_c

$G_4 D$:



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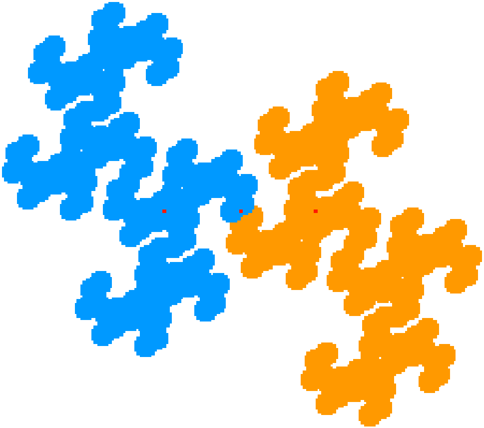
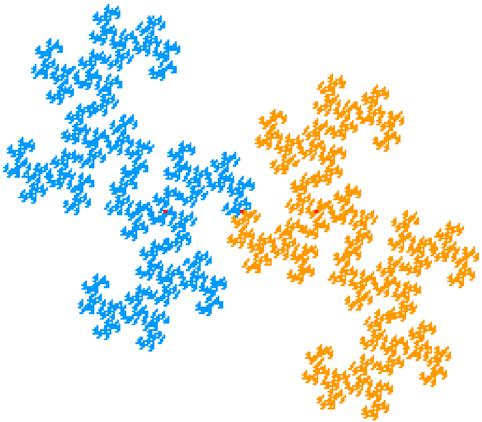
Λ_c

$G_8 D$:



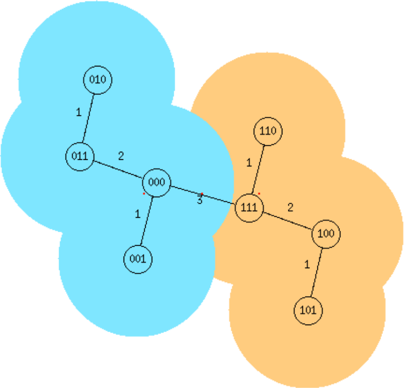
Λ_c

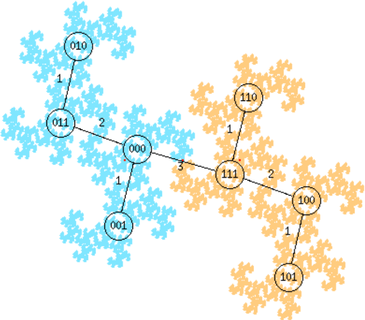
$G_{12}D$:



Λ_c

Consider $G_n D$ (here $G_3 D$). The limit set Λ_c is a union of copies of Λ_c, one in each disk in $G_n D$:





Parameterized IFS

The parameter space for the IFS $\{f_c, g_c\}$ is the open unit disk $\mathbb{D}$. We define:

$$\begin{aligned} \mathcal{M} &= \{c \in \mathbb{D} \,|\, \Lambda_c \text{ is connected}\} \\ \mathcal{M}_0 &= \{c \in \mathbb{D} \,|\, 0 \in \Lambda_c\} \end{aligned}$$

Lemma

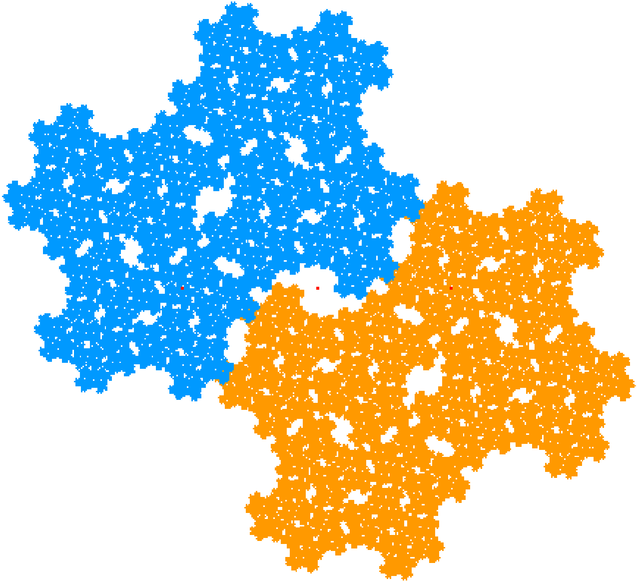
$\mathcal{M}_0 \subsetneq \mathcal{M}$.

Note the distinction with the Mandlebrot set; sets $\mathcal{M}$ and $\mathcal{M}_0$ are *different*.

Lemma (Bandt)

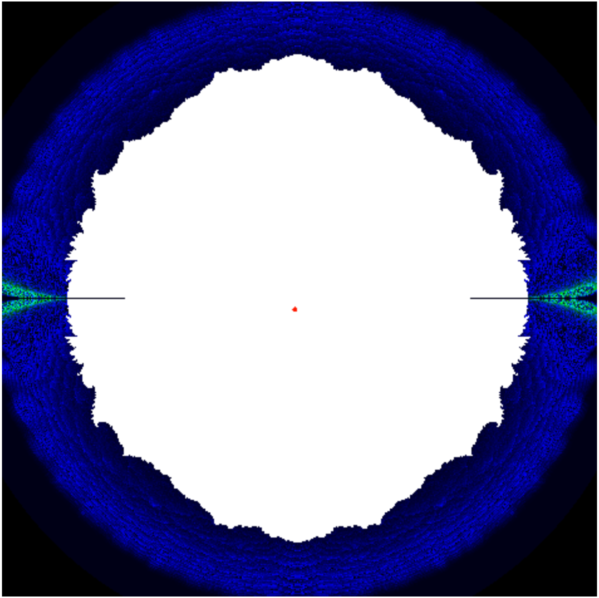
$$\begin{aligned} c \in \mathcal{M} &\Leftrightarrow \Lambda_c \text{ connected} \\ &\Leftrightarrow \Lambda_c \text{ is path connected} \\ &\Leftrightarrow f_c\Lambda_c \cap g_c\Lambda_c \neq \varnothing \end{aligned}$$

$\mathcal{M}_0 \subsetneq \mathcal{M}$

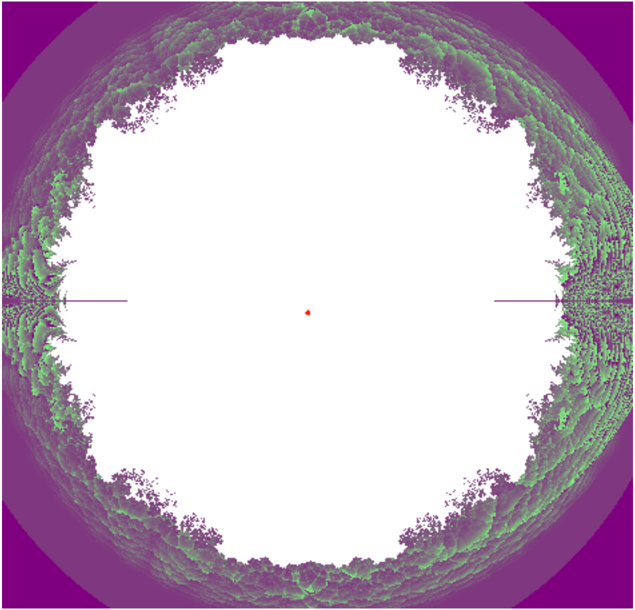


$c = 0.22 + 0.66i$ is in $\mathcal{M}$ but not $\mathcal{M}_0$.

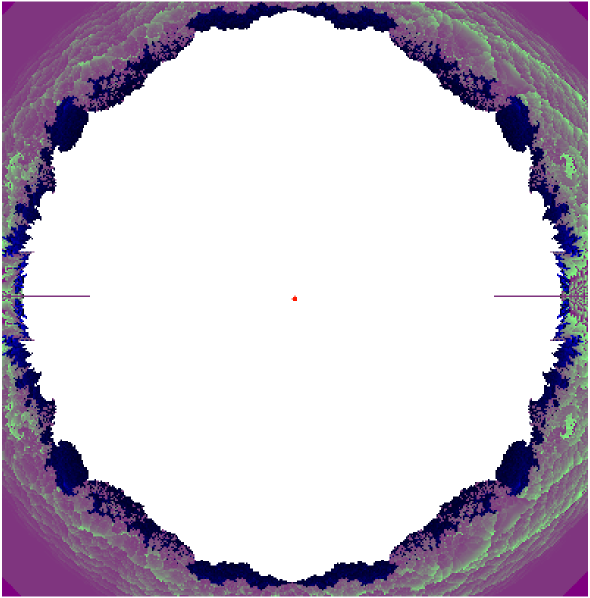
Here is $\mathcal{M}$:



Here is $\mathcal{M}_0$:

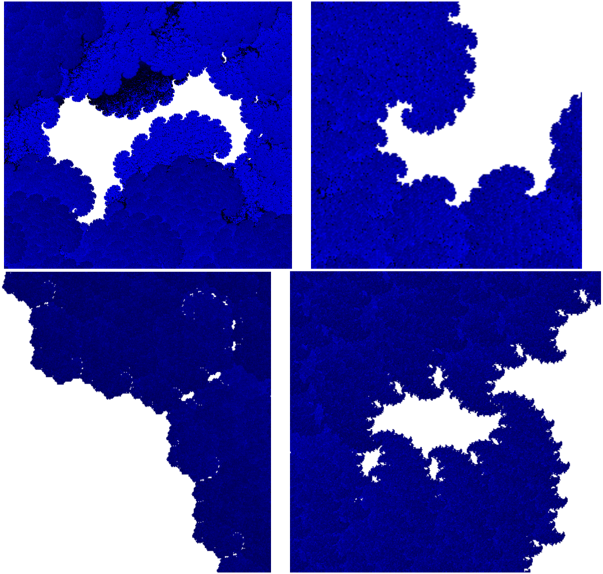


Sets $\mathcal{M}$ and $\mathcal{M}_0$ together:



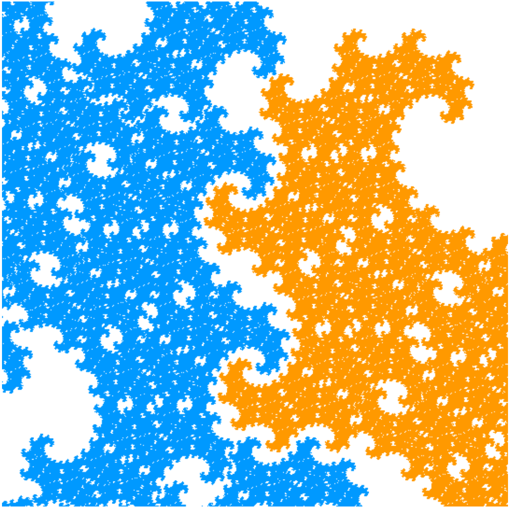
Set $\mathcal{M}$

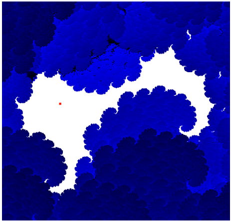
Set $\mathcal{M}$ has many interesting features:



Holes

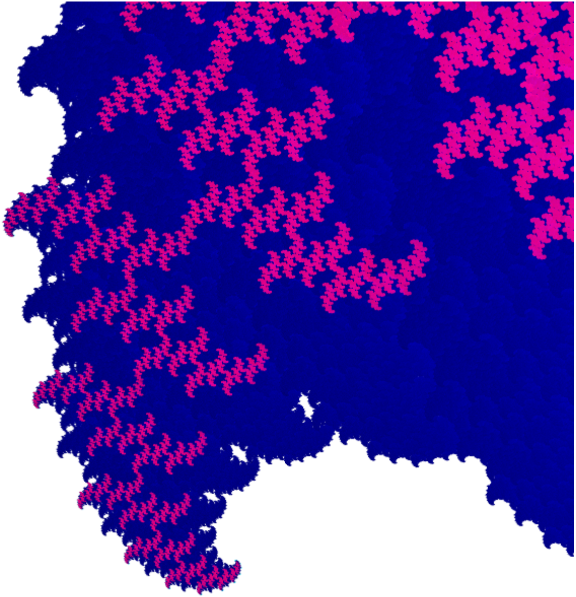
Apparent holes in $\mathcal{M}$ are caused by $f_c\Lambda_c$ and $g_c\Lambda_c$ interlocking but not touching.





(zoomed picture of Λ_c)

Sets $\mathcal{M}$ and $\mathcal{M}_0$



(fun with `schottky`)

History

Barnsley and Harrington (1985) defined sets $\mathcal{M}$ and $\mathcal{M}_0$ and noted apparent holes in $\mathcal{M}$

Bousch 1988 Sets $\mathcal{M}$ and $\mathcal{M}_0$ are connected and locally connected

Odlyzko and Poonen 1993 Zeros of $\{0, 1\}$ polynomials (related to $\mathcal{M}_0$) is connected and path connected

Bandt 2002 Proved a hole in $\mathcal{M}$, conjectured that the interior of $\mathcal{M}$ is dense away from the real axis

Solomyak and Xu 2003 Proved that the interior is dense in a neighborhood of the imaginary axis

Solomyak (several papers) proved interesting properties of $\mathcal{M}$ and $\mathcal{M}_0$, including a self-similarity result.

Thurston 2013 Studied entropies of postcritically finite quadratic maps and produced a picture which, inside the unit disk, appears to be $\mathcal{M}_0$.

Tiozzo 2013 Proved Thurston's set is $\mathcal{M}_0$ (inside the unit disk).

Results

Theorem
There is an algorithm to certify that a point lies in the interior of $\mathcal{M}$

(and consequently to certify holes in $\mathcal{M}$; this is a very different method than Bandt's certification of a hole)

Theorem (Bandt's Conjecture)
The interior of $\mathcal{M}$ *is dense in* $\mathcal{M}$ *away from the real axis.*

Theorem
There is an infinite spiral of holes in $\mathcal{M}$ *around the point* $\omega \approx 0.371859 + 0.519411i$.

There are many infinite spirals of holes, and our method should work for any of them; we just happened to do it for ω.

Properties of Λ_c

Recall:

Lemma (Bousch)

$c \in \mathcal{M}$ (Λ_c is connected) $\Leftrightarrow f_c\Lambda_c \cap g_c\Lambda_c \neq \varnothing$.

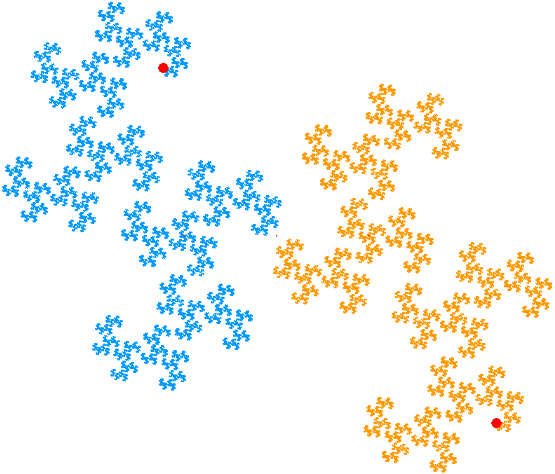
We prove:

Lemma (The short hop lemma - CKW)

If $d(f_c\Lambda_c, g_c\Lambda_c) = \delta$, then for any two points $p, q \in \Lambda_c$, there is a sequence of points $p = s_0, s_1, \ldots, s_k = q$ such that $s_i \in \Lambda_c$ and $d(s_i, s_{i+1}) \leq \delta$.

The short hop lemma

If $d(f_c\Lambda_c, g_c\Lambda_c) = \delta$, then for any two points $p, q \in \Lambda_c$, there is a sequence of points $p = s_0, s_1, \ldots, s_k = q$ such that $s_i \in \Lambda_c$ and $d(s_i, s_{i+1}) \leq \delta$.



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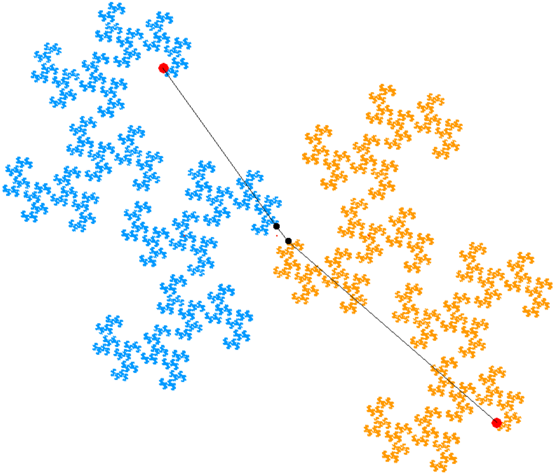

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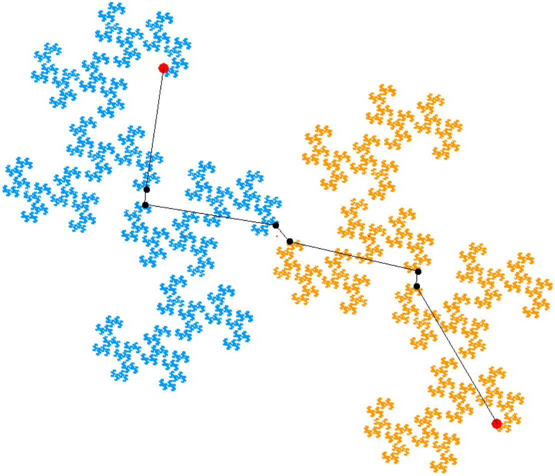

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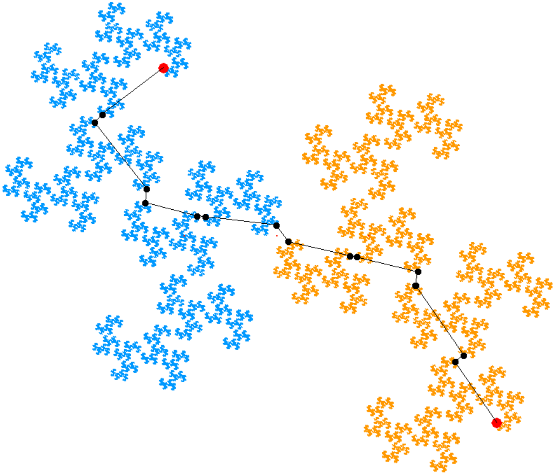

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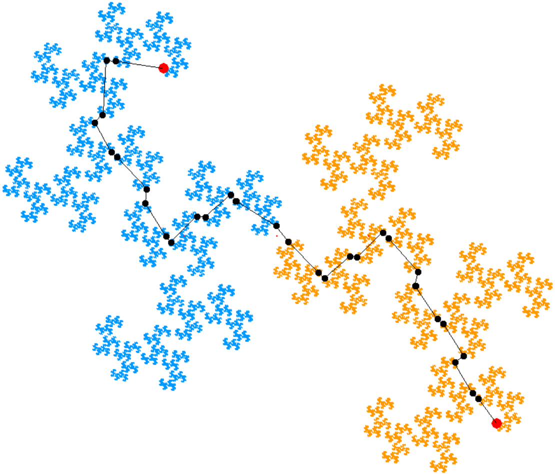

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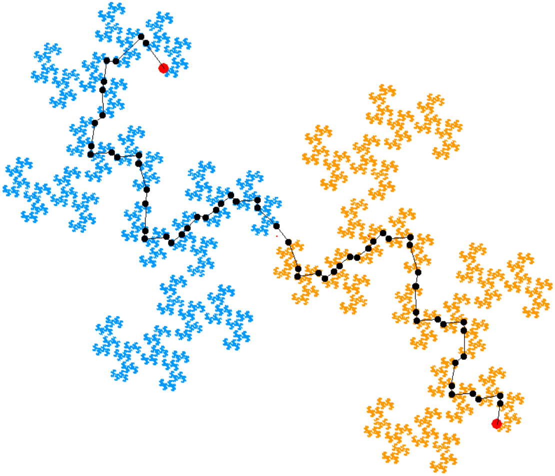


Traps

Theorem

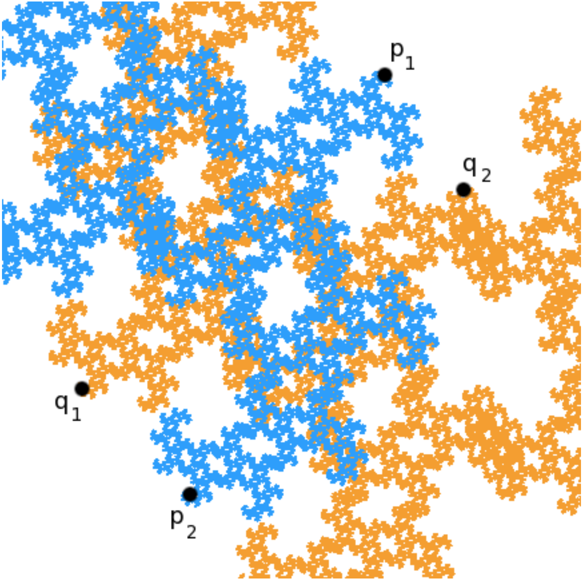
There is an algorithm to certify that a point lies in the interior of $\mathcal{M}$

We'll show there is an easy-to-check condition which certifies that $f_c\Lambda_c \cap g_c\Lambda_c \neq \varnothing$, and thus that Λ_c is connected, and this condition is *open*.

This certificate is called a *trap*.

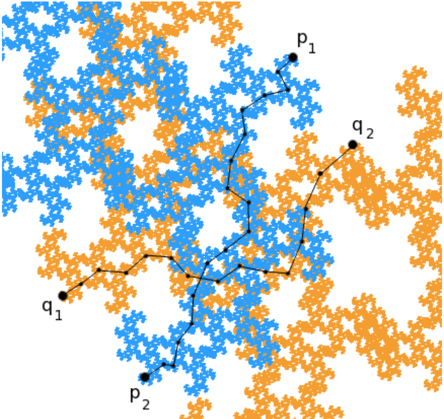
Traps

Suppose that $f_c\Lambda_c$ and $g_c\Lambda_c$ cross "transversely":

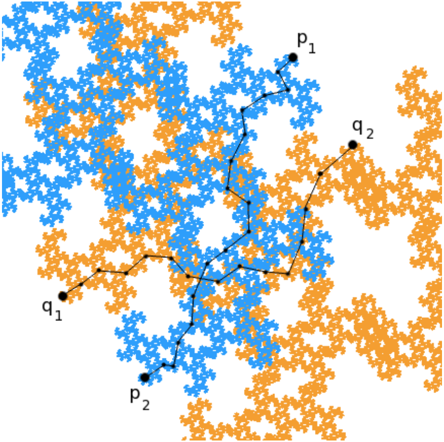


Traps

- Suppose that $f_c\Lambda_c$ and $g_c\Lambda_c$ cross transversely and that $d(f_c\Lambda_c, g_c\Lambda_c) = \delta$.
- By lemma, there are short hop paths $p_1 \to p_2$ in $f_c\Lambda_c$ and $q_1 \to q_2$ in $g_c\Lambda_c$, and these paths have gaps $\leq \delta$.
- The paths cross, so there is a pair of points, one in $f_c\Lambda$ and one in $g_c\Lambda_c$, with distance $< \delta$. A contradiction unless $\delta = 0$.



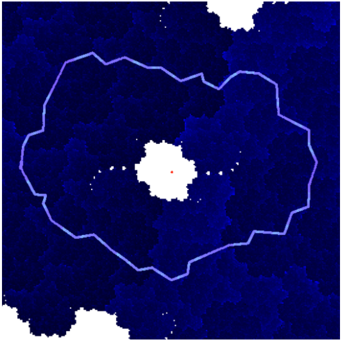
Traps

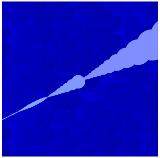


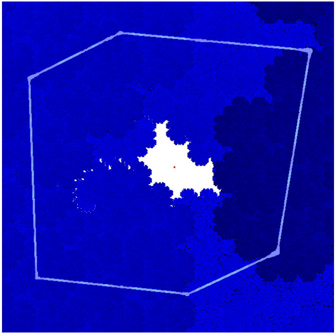
Note the existence of a trap is an open condition, so it certifies a parameter c as being in the *interior* of set $\mathcal{M}$.

Trap loops

Since a trap is an open condition, each trap certifies a small ball as being in $\mathcal{M}$. Using careful estimates, we can surround an apparent hole with these balls to rigorously certify it:

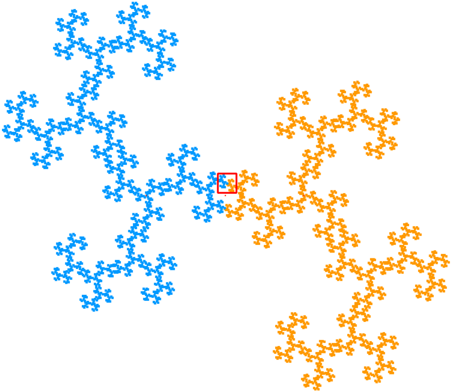






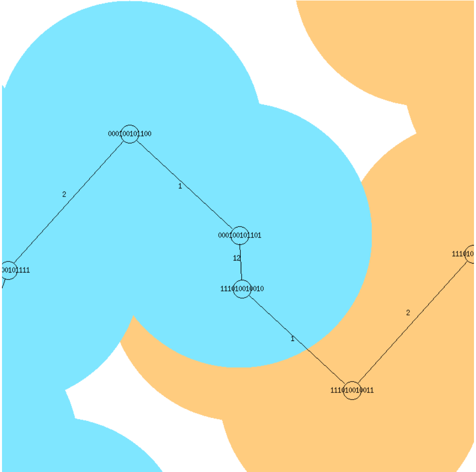
Finding traps

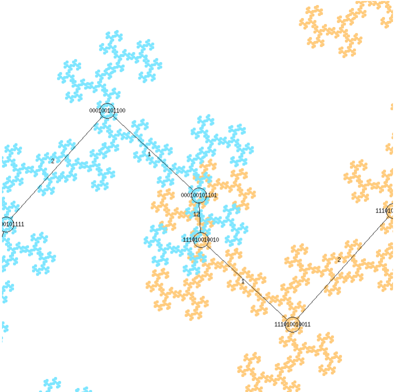
To find a trap, we want to show that $f_c\Lambda_c$ is transverse to $g_c\Lambda_c$. It suffices to find two words u, v starting with f, g such that $u\Lambda_c$ is transverse to $v\Lambda_c$:



Finding traps

It suffices to find two words u, v starting with f, g such that $u\Lambda_c$ is transverse to $v\Lambda_c$. Here $u = fffgffgfggfg$, $v = gggfgffgffgf$





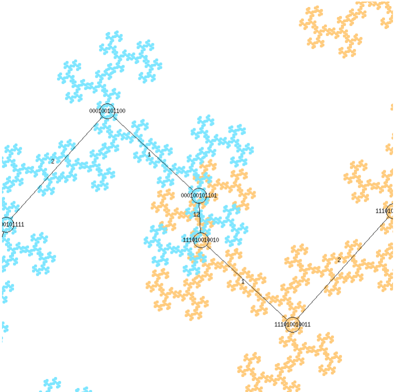
The center of the disk D is 0, so the displacement vector between $u\Lambda_c$ and $v\Lambda_c$ is $u(0) - v(0)$.

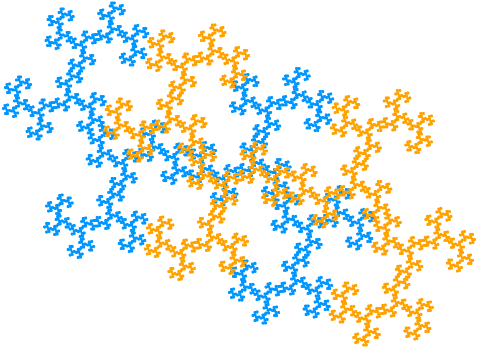
Finding traps

The displacement vector between $u\Lambda_c$ and $v\Lambda_c$ is $u(0) - v(0)$. Note that rescaling the displacement:

$$c^{-12}(u(0) - v(0))$$

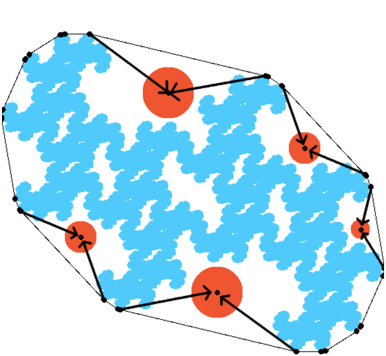
Gives us the displacement vector relative to the original limit set Λ_c:

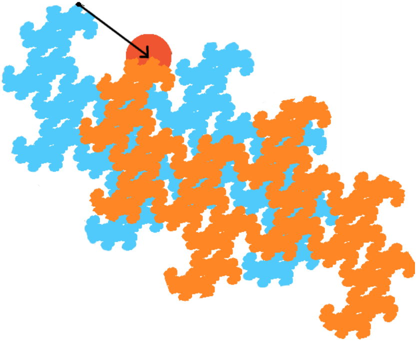




Finding traps

If we consider Λ_c, we can figure out what displacement vectors make it transverse:

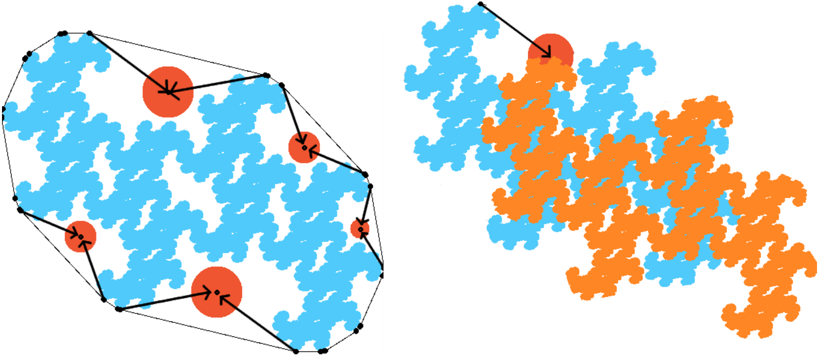




These are *trap-like* vectors for Λ_c. We have shown: if u, v of length n start with f, g, and $c^{-n}(u(0) - v(0))$ is trap-like, then there is a trap for c.

Finding traps

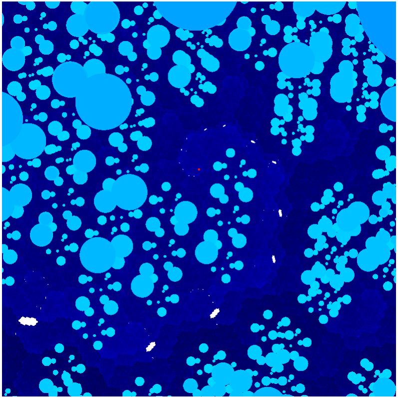
We have shown: if u, v of length n start with f, g, and $c^{-n}(u(0) - v(0))$ is trap-like, then there is a trap for c.

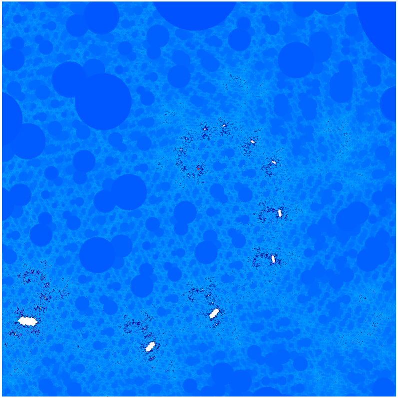


This is computationally useful, because trap-like vectors are trap-like for a whole ball of parameters. To find traps in a region in $\mathcal{M}$, we can find trap-like vectors *once*; then for a given parameter, find words u, v so $c^{-n}(u(0) - v(0))$ is trap-like.

Finding traps

We find trap-like vectors for this (quite small) region in parameter space. Then for every pixel, we search through pairs of words u, v trying to find a pair so $c^{-n}(u(0) - v(0))$ is trap-like.

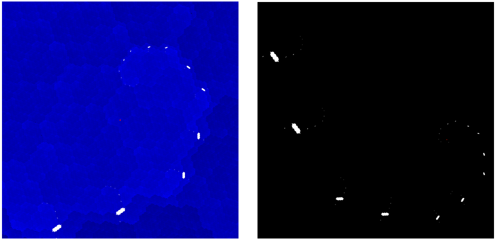




Left, the result of searching words through length 20. Right, through length 35.

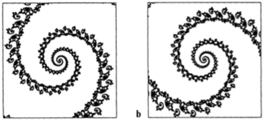
Similarity

On the left is $\mathcal{M}$ near $c = 0.371859 + 0.519411i$. On the right is a (zoomed) view of the set of *differences* between pairs of points in the limit set Λ_c.



This similarity is analogous to the Mandelbrot/Julia set similarity at Misiurewicz points:

Similarity Between the Mandelbrot Set and Julia Sets 611



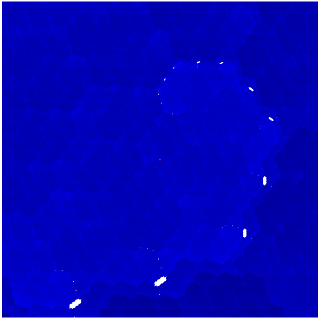
a b

Fig. 9a, b. Magnifications of J_c and of M for $c = -0.77568377 + 0.13646737i$. **a** J_c, center: c, width: 0.00018. **b** M, center: c, width: 0.00024

(picture by Tan Lei)

Similarity

On the left is $\mathcal{M}$ near the parameter $0.371859 + 0.519411i$. On the right is a (zoomed) view of the set of *differences* between pairs of points in the limit set Λ_c.



Theorem (Solomyak)

These sets are asymptotically similar. (Small neighborhoods Hausdorff converge).

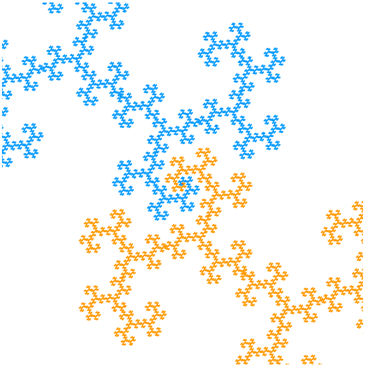
We can re-prove this theorem (with a bonus: asymptotic interior!) using traps.

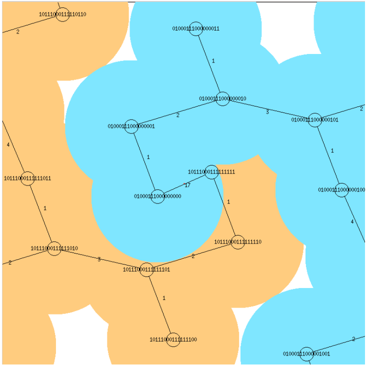
Similarity

For parameters near $\omega \approx 0.371859 + 0.519411i$, the only possible pairs of words that give traps are of the form

$$u = fgfffgggf^n x, \qquad v = gfgggfffg^n y$$

for large n

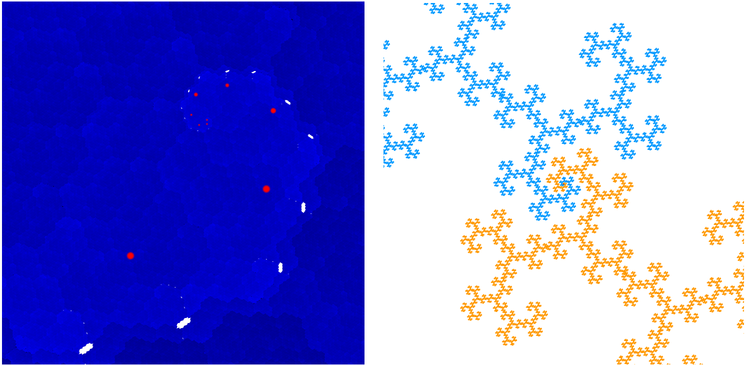




The definition of ω is that it is the parameter such that $fgfffgggf^\infty = gfgggfffg^\infty$.

Similarity

The dots show $\omega + C$, $\omega + C\omega$, $\omega + C\omega^2$...



Intuitively, moving from $\omega + C\omega^k$ to $\omega + C\omega^{k+1}$ should correspond to zooming in on the spiral in the limit set. The orientation of the disks $fgfffgggf^n$, $gfgggfffg^n$ at parameter $\omega + C\omega$ should look like the orientation of the disks $fgfffgggf^{n+1}$, $gfgggfffg^{n+1}$ at $\omega + C\omega^2$.

Similarity

Lemma

Suppose $uf^\infty = vg^\infty$ for parameter ω. Let u, v have length a. Let x, y have length c. As $n \to \infty$, the expression

$$(\omega + C\omega^n)^{-a+n+c}(uf^n x(0) - vg^n y(0))$$

converges to

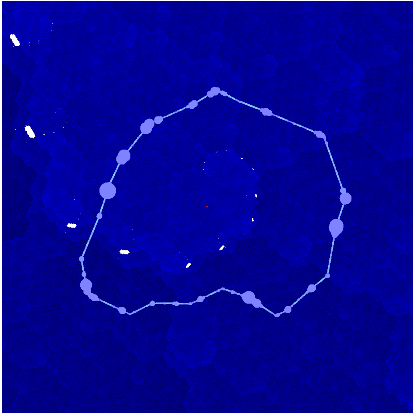
$$V = \omega^{-a-c}(u(0) - v(0)) + \omega^{-c}(x(0) - y(0)) + C\omega^{-a-c}P'(\omega)$$

(where $P'(\omega)$ is a constant depending only on ω).

Hence if V is trap-like for ω, then for all sufficiently large n, the words $uf^n x$,$vg^n y$ give a trap for $\omega + C\omega^n$. We call this a *limit trap*.

Note there is *one* computation required to prove *infinitely* many points have traps (are in the interior of $\mathcal{M}$).

Infinitely many holes in $\mathcal{M}$



Here we found a loop of balls of limit traps. This certifies that for sufficiently large n, the image of this loop under the map $z \mapsto \omega(z - \omega) + \omega$ lies in the interior of $\mathcal{M}$.
We also prove there are points limiting to ω in the complement of $\mathcal{M}$. Together, this proves infinitely many holes.

What does this have to do with differences?

If C is such that

$$V = \omega^{-a-c}(u(0) - v(0)) + \omega^{-c}(x(0) - y(0)) + C\omega^{-a-c}P'(\omega)$$

is trap-like, then $\omega + C\omega^n$ have traps. Let's solve for the C values which work. Let V range over all trap-like vectors; we get C values:

$$C = \frac{\omega^{a+c}}{P'(\omega)}V - \frac{1}{P'(\omega)}(u(0) - v(0)) - \frac{\omega^a}{P'(\omega)}(x(0) - y(0))$$

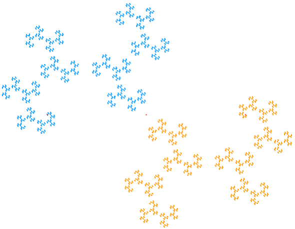
As $c \to \infty$, the first term $\to 0$, and the second two become a scaled, translated copy of Γ_ω, the set of all *differences* between points in Λ_ω, i.e.

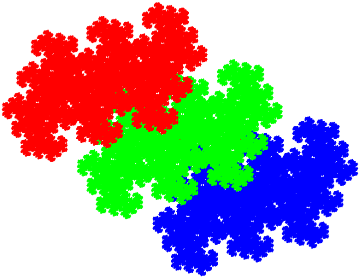
$$C \in A + B\Gamma_\omega$$

Differences as limit sets

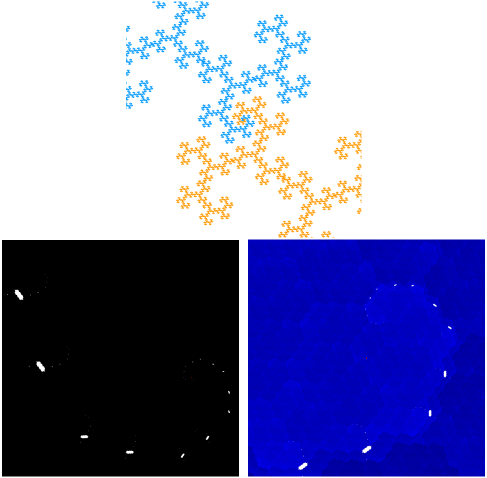
The set of differences Γ_c between points in Λ_c is a limit set itself!
It is the limit set of the three-generator IFS:

$$z \mapsto cz - 1, \qquad z \mapsto cz, \qquad z \mapsto cz + 1$$



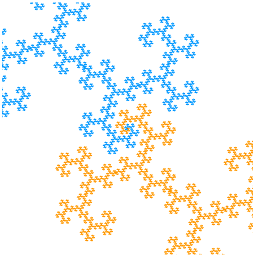


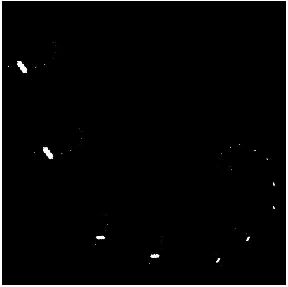
Hence:

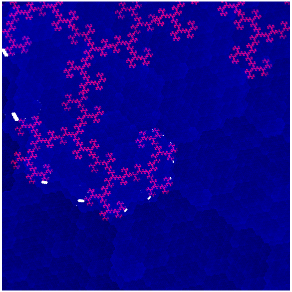


The differences Γ_ω of points in the limit set Λ_ω (top) is a limit set itself (bottom left), and Γ_ω is locally the same as $\mathcal{M}$ in a neighborhood of ω (bottom right).

Hence:







We can't help remarking that in addition, set $\mathcal{M}_0$ looks locally like Λ_ω!

More pictures!