Stable immersions in orbifolds

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Question

Given a map f from a surface with boundary into a hyperbolic orbifold O with boundary, when is f homotopic to an immersion with geodesic boundary?

Question

Given an element $w \in \pi_1(O)$, when is there an immersion with geodesic boundary such that the boundary is a loop representing w?

Stability

Theorem (W)

Let O be a hyperbolic orbifold with genus ≥ 1 and one boundary component b, and let $w \in \pi_1(O)$. Then for sufficiently large N, some power of wb^N bounds an immersed surface with geodesic boundary.

If the orbifold is a disk, then it holds for any sufficiently large N divisible by $\gcd(o_0, \cdots o_n)$, where the o_i are the orbifold point orders.

Compare to:

Theorem (Calegari-Louwsma)

The above theorem for $(2, p, \infty)$ orbifolds.

Theorem (Calegari)

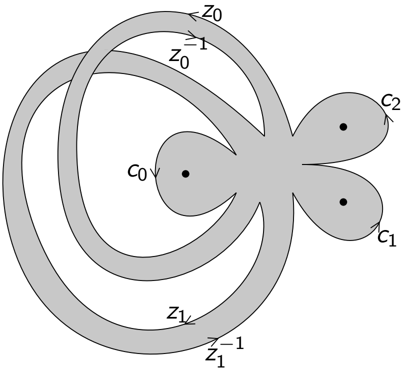
The above theorem for $w + b^N$.

Combinatorial surface maps

We can write $\pi_1(O)$ as generated by some infinite-order generators and some finite-order generators:

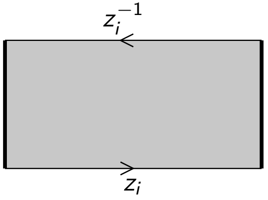
$$\begin{aligned}\pi_1(O) &= \mathbb{Z} * \cdots * \mathbb{Z} * \mathbb{Z}/o_0 * \cdots * \mathbb{Z}/o_m \\ &= \langle z_0, z_1, \cdots, z_k, c_0, c_1, \cdots, c_m \mid c_i^{o_i} \rangle\end{aligned}$$

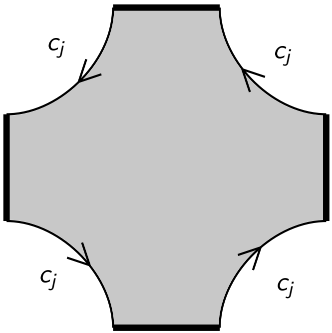
To be concrete, set $\pi_1(O) = \langle z_0, z_1, c_0, c_1, c_2 \mid c_0^4 = c_1^4 = c_2^4 = \mathrm{id} \rangle$
Here is a picture of the orbifold O.



Combinatorial surface maps

Consider the surface pieces below

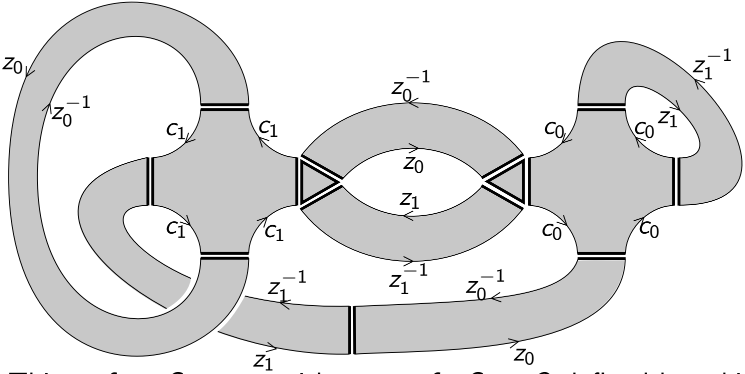




Suppose we glue up some of these *rectangles* and *group polygons* into a surface with reduced boundary...

Combinatorial surface maps

Suppose we glue up some of these pieces into a surface with reduced boundary:

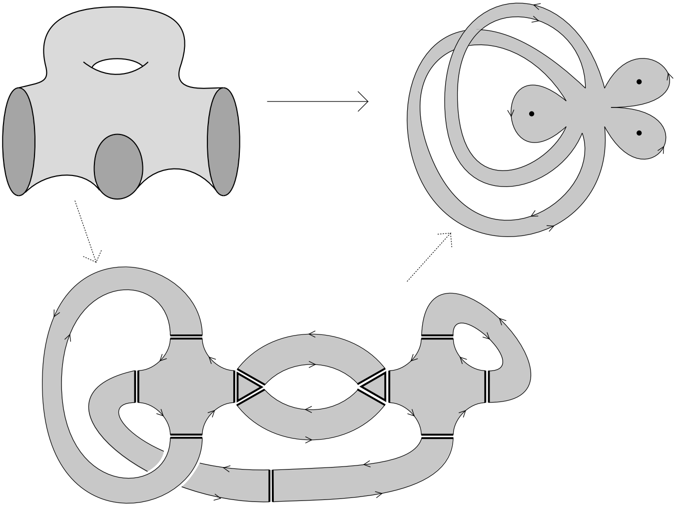


This surface S comes with a map $f : S \to O$ defined by taking the junctions (*polygons*) to the base point, and taking the rectangles and group polygons around the loops and orbifold points. This is a *fatgraph map*.

Combinatorial surface maps

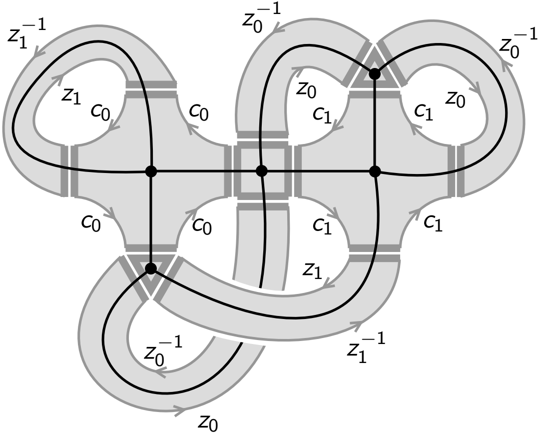
Proposition

Every surface map to an orbifold with boundary factors through a fatgraph map, possibly after compression.



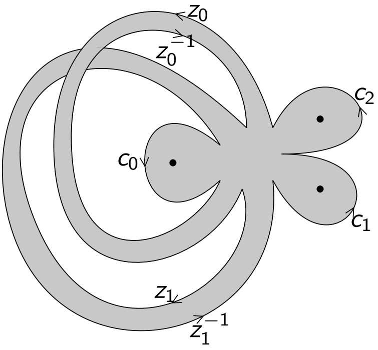
Combinatorial surface maps

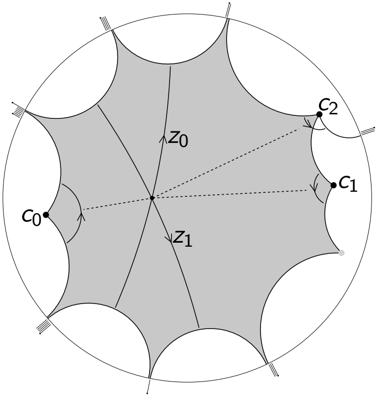
A fatgraph has a *spine* to which it deformation retracts:



Hyperbolic structure

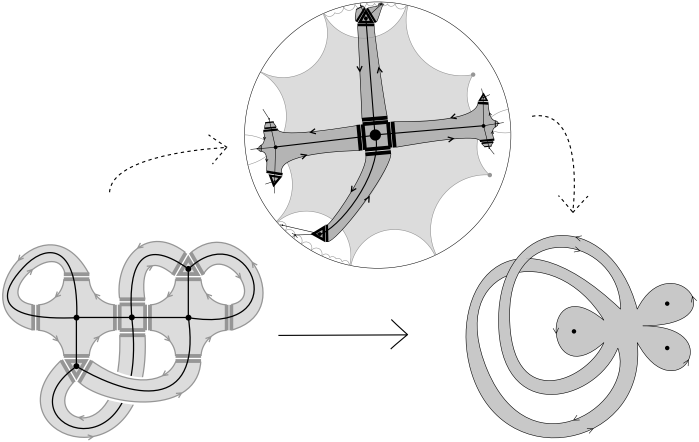
The hyperbolic structure on the orbifold identifies its universal cover with the hyperbolic plane (let the boundaries by parabolic):





Spines of fatgraph maps

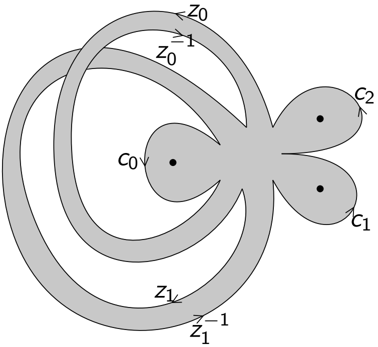
Thus, a surface map into the orbifold maps the universal cover of the surface into the hyperblic plane:

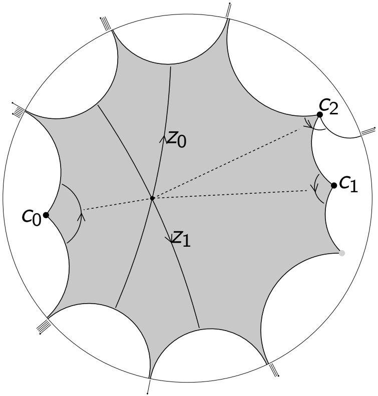


The fatgraph map is homotopic to an immersion with geodesic boundary iff the spine lays out correctly (preserving cyclic orders on the vertices) in the universal cover.

Cyclic orders

A hyperbolic structure on the orbifold determines a cyclic order on the generators of the fundamental group

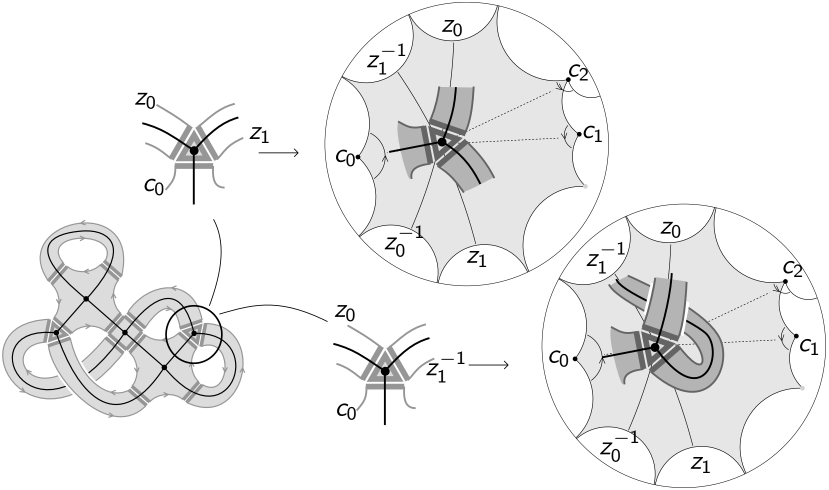




This orbifold has cyclic order $[c_1, c_2, z_0, z_1^{-1}, c_0, z_0^{-1}, z_1]$.

Spines of fatgraph maps

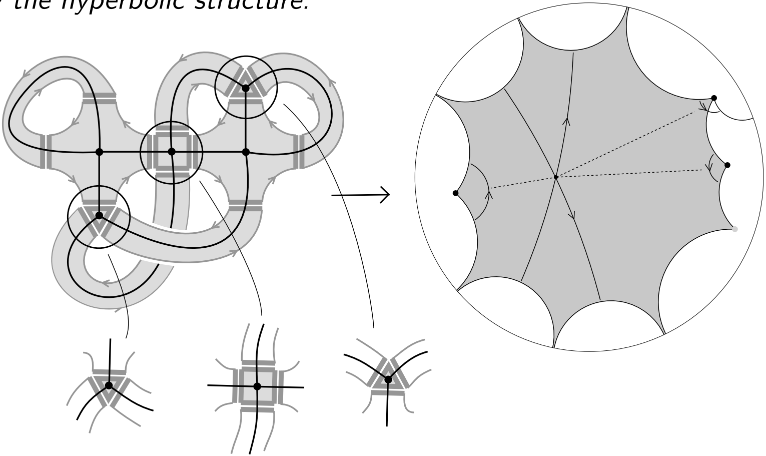
To determine if the spine embeds in the universal cover preserving orientation, it suffices to check the cyclic orders on the polygons.



Cyclic orders and immersions

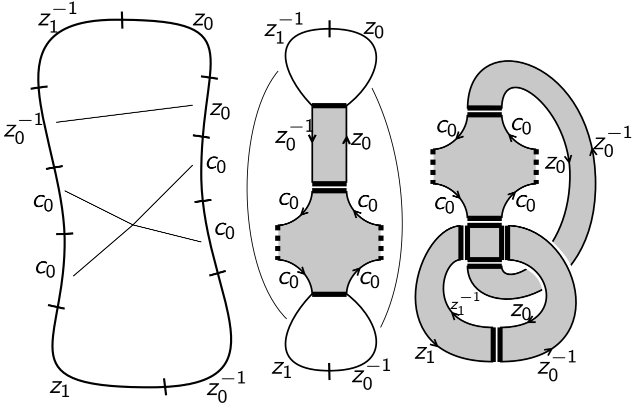
Lemma

A fatgraph map is homotopic to an immersion with geodesic boundary iff the cyclic orders on the outgoing edges of each polygon are compatible with the cyclic order on generators induced by the hyperbolic structure.



Building immersed surfaces

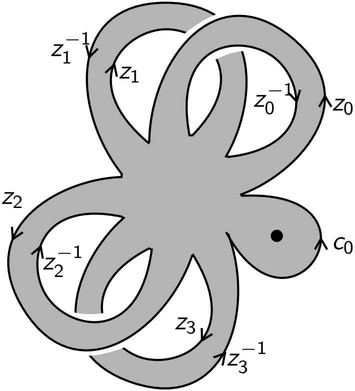
Thus, to show that a given word (or words) bounds a surface map with geodesic boundary, it suffices to find a combinatorial surface with the desired boundary such that each polygon is compatible with the orbifold order.

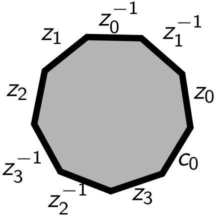


Can we pinch a loop together in such a way that the polygons are all compatible with the cyclic order?

Building immersed surfaces

Given $w \in \pi_1(O)$ arbitrary, we want to build a surface with boundary wb^N, where b is the boundary of O.
Say O is a genus two orbifold with one orbifold point, and the indicated cyclic order:

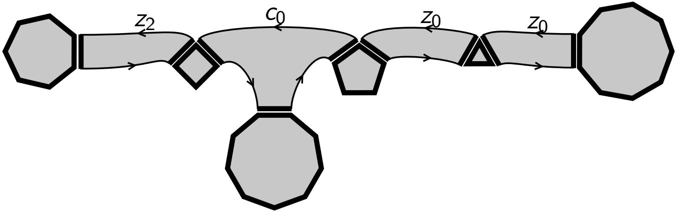




Note there is a maximal compatible polygon (right).

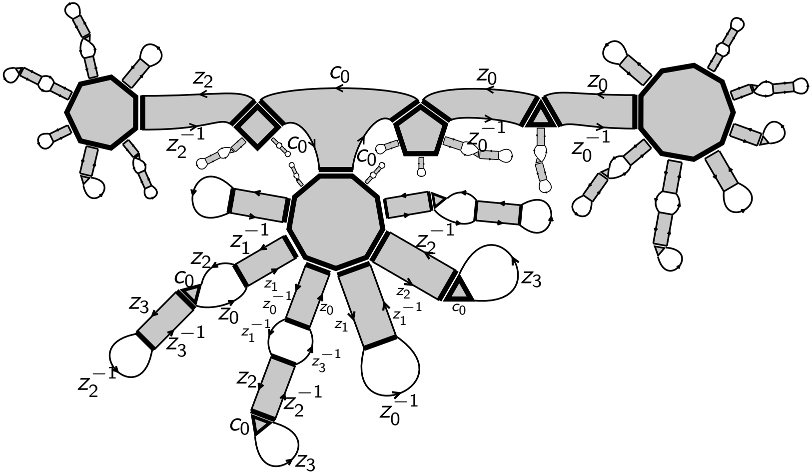
Building immersed surfaces

Start with a partial fatgraph whose boundary contains the desired word:



Building immersed surfaces

Then attach lots of standard bits which produce boundary with many copies of the boundary word:



Fact: the unglued part of the boundary glues up.