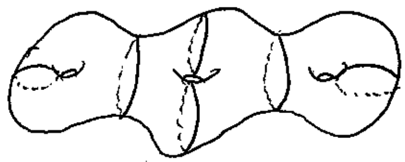
# Finding Surface Subgroups 

Theme
- Where do surfaces appear in other things?
- What can they do?
- How can we find/build them?

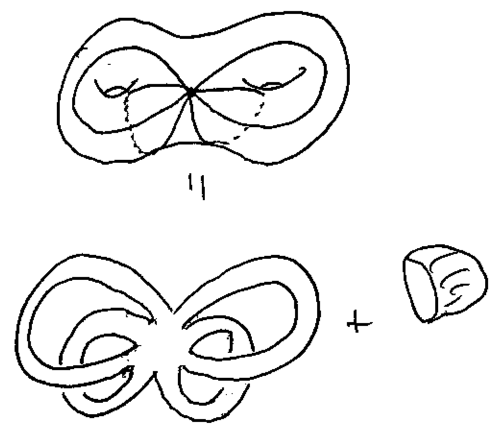
We can build/think of surfaces in many ways
we'll see all of them in action

e.g. from cells

from pants



from a spine



Recall Surface complexity
Euler characteristic $\chi(S) =$

$2 - 2g - p$

genus, punctures/boundaries

$= \underset{\text{vertices}}{v} - \underset{\text{edges}}{e} + \underset{\text{faces}}{f}$ in cellulation

$= -(\#\text{ of pants})$

$= \underset{\text{vertices}}{v} - \underset{\text{edges}}{e} + \text{disks}$ (in spine)

# Normal Surfaces

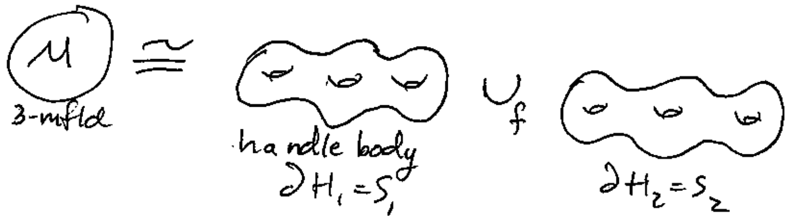


## Recall

Every 3-manifold can be triangulated
(Rmk: not simple, and not true for 4-manifolds)

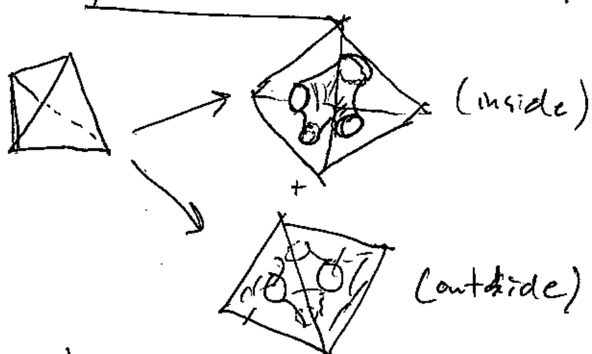
## Rmk

this immediately shows that every 3-manifold has a Heegaard splitting:




where $f : S_1 \to S_2$ is a homeo

Ex $S^3$

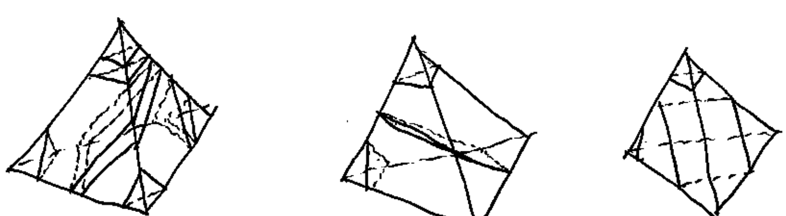




both deformation retract to graphs, so are handlebodies

which connects the mapping class groups of surfaces to 3-manifolds

## Def

A Normal surface is an embedded surface in a triangulated 3-mfld which intersects each tetrahedron in triagles + quads



note there are 3 kinds of quads in each tetrahedad, only one of which can occur.

**Rmk**

Normal surfaces solve many problems for 3-manifolds (ie answer many questions about specific manifolds)

**Recall** (Kneser)

$\mathcal{M}$ 3mfld, then $\mathcal{M} = \mathcal{M}_1 \underset{\text{connect sum}}{\#} \mathcal{M}_2 \# \cdots \# \mathcal{M}_n$,

where $\mathcal{M}_i$ are prime

$$\left.\begin{array}{l}\text{cannot be}\\ \text{written as a}\\ \text{connect sum}\end{array}\right\} = \begin{array}{c} S^1 \times S^2 \\ \text{or} \\ \underline{\text{irreducible}} \\ \text{every 2sphere bounds a ball}\end{array}$$

the decomposition is unique (up to $S^1 \times S^2$'s)

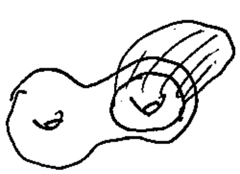
**Thm (Kneser)**

There is a decomposition of $\mathcal{M}$ in which the decomposition is given by an embedded normal collection of 2-spheres

**Recall**

$S$ surface embedded in 3-mfld $\mathcal{M}$.

$S$ is incompressible if there is no disk $D \subseteq \mathcal{M}$ embedded s.t. $\partial D \subseteq S$ doesn't bound a disk in $S$:



**Thm (Dehn's lemma / loop thm)**

If $S \subseteq \mathcal{M}$ 2-sided, then $S$ incompressible $\Leftrightarrow$ $\pi_1(S) \hookrightarrow \pi_1(\mathcal{M})$ injective

**Rmk** (simple loop conjecture) (open)

Given any map $S \xrightarrow{f} \mathcal{M}$, we have $f_*: \pi_1(S) \to \pi_1(\mathcal{M})$ injective $\Leftrightarrow$ there is no simple loop (embedded in $S$) in $\ker(f_*)$

**Def**

3-mfld is **Haken** if it is irreducible and contains a 2-sided ~~irreducible~~ incompressible surface

**Thm (Haken)**

An ~~[illegible]~~ incompressible surface is isotopic to a normal one.

**Rmk**

Finding normal surfaces can be done computationally with linear programming (although not polynomial, because we must branch on the quads (i.e. actually integer programming))

**Regina** (software by many people, see Ben Burton)

Does (among other stuff):

- 3-sphere, 3-ball recognition
- irreducibility + connect sum decomp.
- Hakenness testing

I.e. normal surfaces compute the answers to many questions about specific manifolds.

What about: is every 3-mfld Haken, etc?

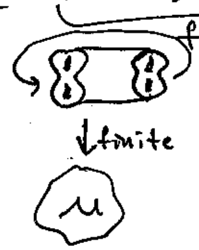
# Surfaces from pants

We **sketch** the proof of:

**Thm** (Kahn-Markovic)
$M$ a closed hyperbolic 3-manifold. There exists a closed hyperbolic surface $S$ and a $(1+\varepsilon)$-quasigeodesic map $f: S \to M$.
(in particular, $f_*: \pi_1(S) \to \pi_1(M)$ is injective)

**Rmk**

- $f$ is not an embedding, hence $M$ need not be Haken
- **Thm** (Agol, Wise, ...)
  "has a finite cover which is"
  Every hyp 3-mfld is virtually Haken, and virtually fibers over the circle.
  ie every hyp 3-mfld




$f$ a surface homeomorphism.

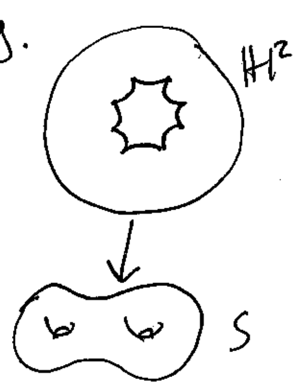
Step 0 is the existence of the $\pi_1$-injective surface (Kahn-Markovic thm)

# Hyperbolicity

A manifold is hyperbolic if it is $\cong \mathbb{H}^n / \Gamma$

hyperbolic n-space.
discrete torsion free group of isometries

e.g.



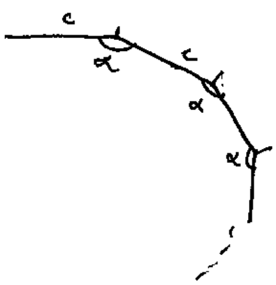
# Features of hyperbolic space



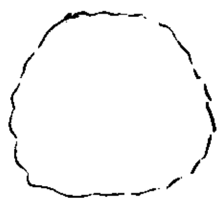
- triangles are thin: $\exists\, \delta > 0$ so for triangle with sides $a, b, c$, $a \subseteq \mathcal{N}_\delta(b) \cup \mathcal{N}_\delta(c)$

- random walks make progress
  or
  the following path never loops:

> Rmk:
> thus slightly bending paths in hyp. manifolds cannot be nullhomotopic




in $\mathbb{R}^2$

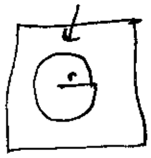


in $\mathbb{H}^2$:

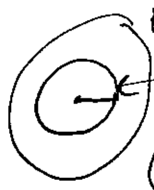
- balls of radius $r$ have <u>more</u> area than in $\mathbb{R}^2$

on $S^2$: less than $\pi r^2$


$\pi r^2$




$\mathbb{H}^2$



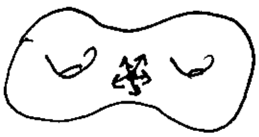
more than $\pi r^2$

$\frac{e^r + e^{-r}}{2}$

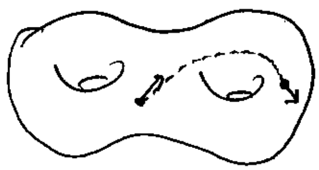
(actually, $= 2\pi(\cosh r - 1)$, so exponential)

# Immersing surfaces in hyp! 3-manifolds

$M$ hyp manifold. the unit tangent bundle $UTM$ is the circle bundle over $M$ of unit tangents



the geodesic flow on $UTM$ moves $(x, v)$ along the geodesic through $x$ in the direction $v$



**Thm** (Brin, Gromov, Moore?)
the geodesic flow on hyperbolic manifolds is exponentially mixing.
(formally, if $f, g$ two functions on $UTM$, $\int_{UTM} f\, \phi_t^* g \to \int f \int g \pm O(e^{-kt})$)
(heuristically most points in $UTM$ will go everywhere all the time

)

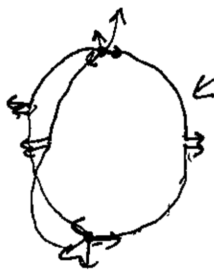
## Finding pants

- pick a point in $M$ and a vector $v$.


- flow $v$ along 3 directions for distance $R$

pick another point and flow:

by changing direction an extremely small amount, we can make the tripods "match"




straighten the curves: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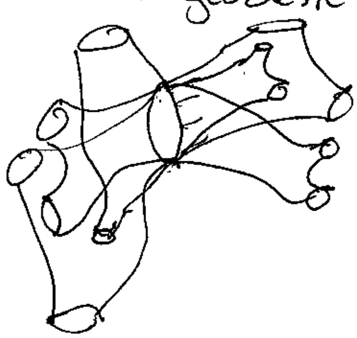
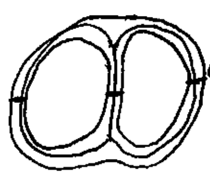
By picking tripods all over $M$ and joining them, we produce many pairs of pants, which all have cuffs of length $2R \pm \varepsilon$

Facts

- there are finitely many geodesics of length $2R \pm \varepsilon$
- each geodesic has many pairs of pants attached to it



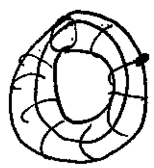
- each p.o.p. marks a distinguished point on cuffs



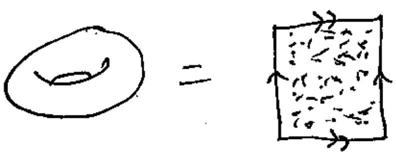

consider geodesic $\gamma$

note each pants marks $\gamma$ in two places with vectors, and $v_2$ is translate of $v_1$, so need only $v_1$.

$\Rightarrow$ each pant is given by a point in (foot) unit tangent bundle of $\gamma$ (a torus $T$)

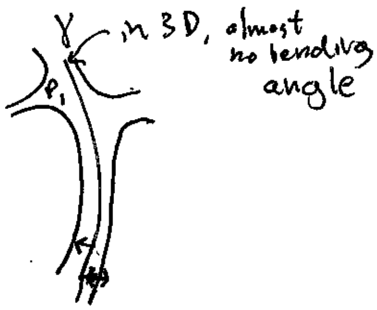


· From mixing of geodesic flow, feet from pants are equidistributed in $T$

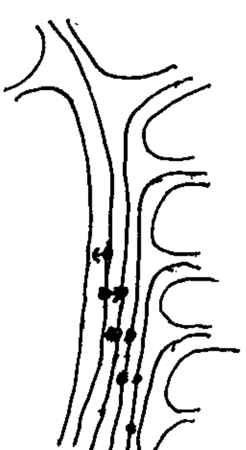


there is a matching between feet to glue each so they are: · translated by 1
· twisted by $\pi \pm \varepsilon$

i.e.




a bunch



Because of the twist:
(side view)


ie geodesics in $S$, glued up pants are piecewise geodesics in $M$ with lengthy segments and small bending angle — cannot be nullhomotopic

so $S \xrightarrow{f} M \rightarrow f_* : \pi_1(S) \hookrightarrow \pi_1(M)$

We find normal surfaces and immersed surfaces in 3-mflds, and they are very useful.
Where else can we find surfaces?

Q] Given a CW complex $X$, can we find a surface map $S \xrightarrow{f} X$ which is $\pi_1$-injective?
(Does $\pi_1(X)$ contain a subgroup $\cong \pi_1(\text{surface})$?)

Q] (Gromov)
Does every 1-ended hyperbolic group contain a surface subgroup?

Hyperbolic groups

$G$ finitely presented

$G = \pi_1(X)$ $\quad X$ CW or simplicial complex

$G$ hyperbolic if universal cover $\tilde{X}$ is $\underbrace{\text{negatively curved}}_{\text{triangles are thin}}$

Ex random one relator groups

$\langle a, b \mid \langle \text{long random word } r \text{ in } a, b \rangle \rangle$

$X =$ 

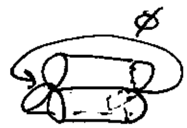
cover  (many many disks at each point)

Ex HNN extensions

Recall mapping tori:

$X =$ rose  $\qquad \pi_1(X) =$ free group $F$

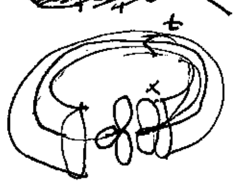
$\phi : X \to X \qquad \phi_* : F \to F$

$X_\phi = \text{rose} \times [0,1] / \phi =$ 

$\pi_1(X_\phi) = F_\phi =$ HNN extension of $F$ by $\phi$

Nice examples of 1-ended hyperbolic groups (for most $\phi$)

$$F_\phi = \langle F, t \mid \forall x \in F,\ t x t^{-1} = \phi(x) \rangle$$



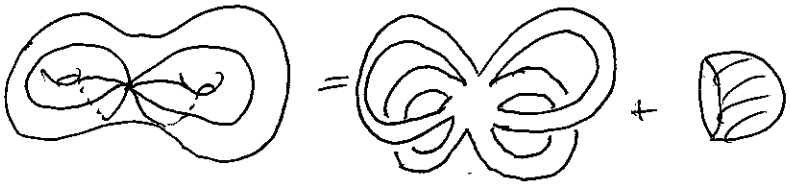
Thm (Calegari–W)

- there is a combinatorial certificate showing the existence of a surface subgroup of $F_\phi$. (for specific $F_\phi$)
- Random HNN extensions contain surface subgroups (many)

First we introduce surfaces from spines and maps into graphs.

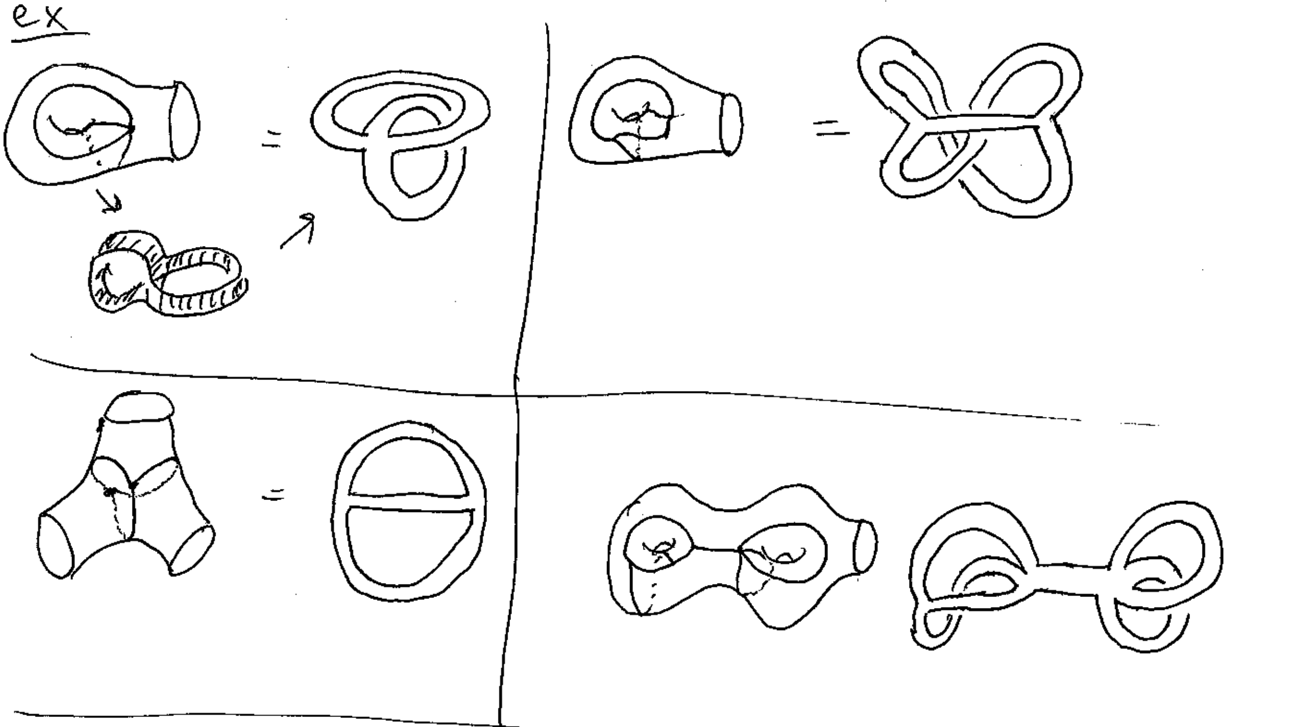
# Surfaces from spines

## Recall



Note a surface with boundary
is homotopic to its spine ← embedded graph which is def. retract of surface
(and ~~...~~ there are many
spines)

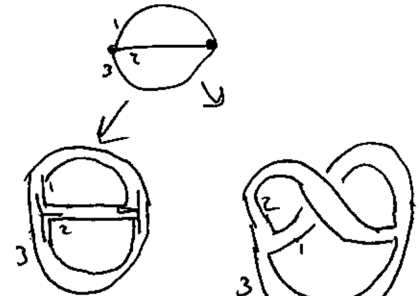
ex



It is convenient to

1) just start with the spine

2) the spine should remember
the cyclic order of edges
at each vertex (i.e. its embedding)




## Def

A graph with a cyclic edge order at each vertex
is a fatgraph / ribbon graph

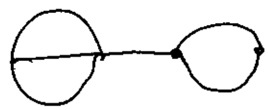
# Surface maps into free groups



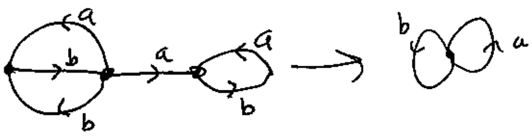
**Ex**

given rose; $X =$ 

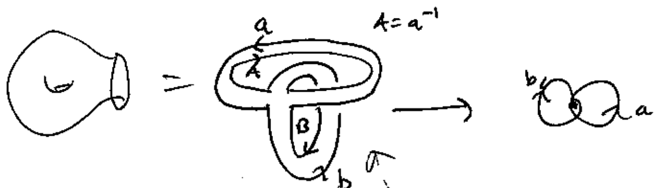

and graph:

$G =$ 

We can specify a map $G \to X$ by labeling $G$:




The same is true for fatgraphs:




record both directions for convenience

**Rmk**

it is easy to read off ~~the~~ the image free group word of $\partial S$ abAB.

**Thm (Culler)**

Every map of a surface with boundary into a rose factors through a labeled fatgraph