Topologically minimal surface maps

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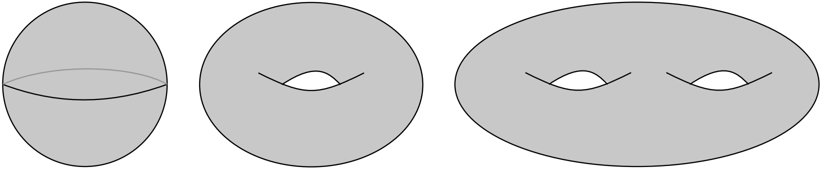
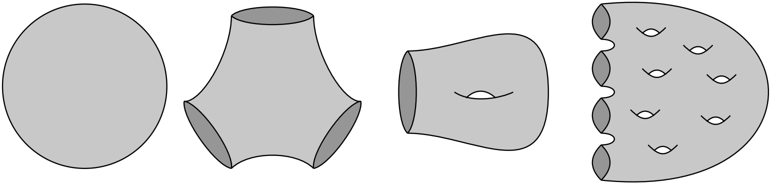
Surfaces

Some surfaces:



Some surfaces with boundary:

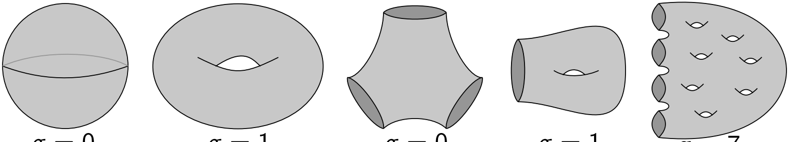


The *genus* is the number of holes.

Euler characteristic

Euler characteristic $\chi(S)$ measures the complexity of S.

$\chi(S) = 2 - 2(\text{genus}) - (\#\text{ boundaries})$



$g = 0$	$g = 1$	$g = 0$	$g = 1$	$g = 7$
$\#b = 0$	$\#b = 0$	$\#b = 3$	$\#b = 1$	$\#b = 4$
$\chi = 2$	$\chi = 0$	$\chi = -1$	$\chi = -1$	$\chi = -16$

Euler characteristic

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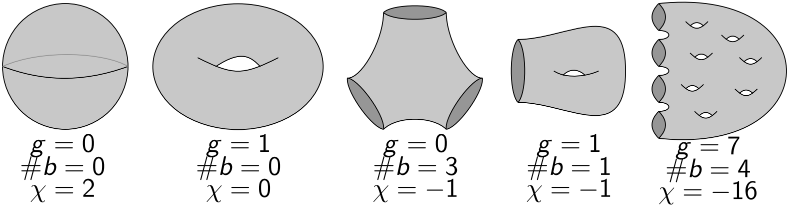
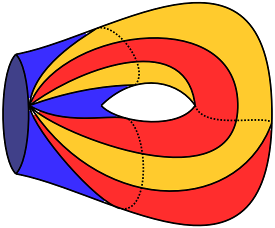


In (any) triangulation of S, $\chi(S) = V - E + F$:



$V - E + F = 1 - 5 + 3 = -1$

Surface maps

Given X topological space, and γ a loop in X, find a surface S and a map $f : S \to X$ so $\partial S \to \gamma$.

I.e. given a loop, find a surface that bounds it.

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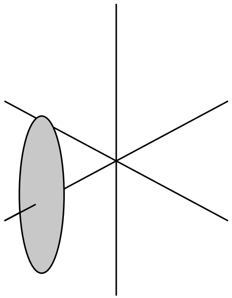
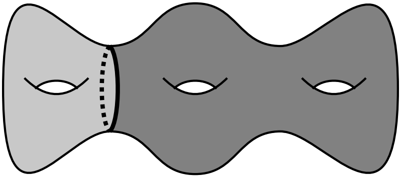
In $\mathbb{R}^3$, any loop bounds a disk:



Surface maps

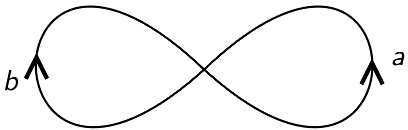
On a surface, some loops *don't* bound disks; the simplest surfaces with them as boundary have higher genus:



There is a genus 1 surface and a genus 2 surface with this boundary loop. (But no surfaces of lower genus).

Simple spaces

A wedge X of two loops:

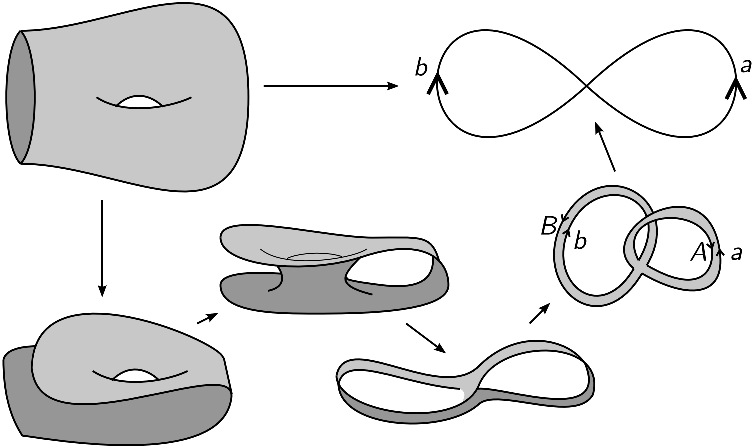


Each loop has a direction labeled, a means forward, A means backward. Loops are recorded by a sequence of letters, e.g. $abAB$.

(The fundamental group (group of loops) is a *free group*).

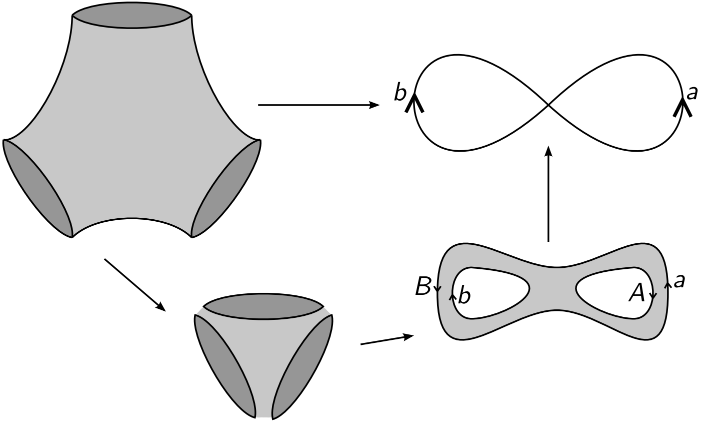
Surfaces in a wedge of loops

Find a surface in X with boundary loop $abAB$. How can a surface map to a wedge of two loops?
Stretch the surface to make it skinny:



Surfaces in a wedge of loops

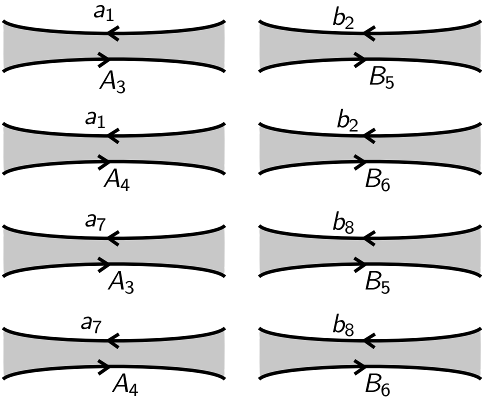
We could ask for a surface with multiple boundary loops:



The skinny surface has boundary $aB + b + A$.

Surfaces in a wedge of loops

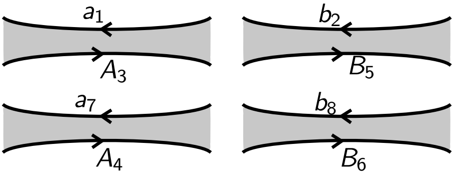
Let us look for a skinny surface with boundary $abAABB + ab$. The strips that can occur are labeled with a letter-inverse pair.

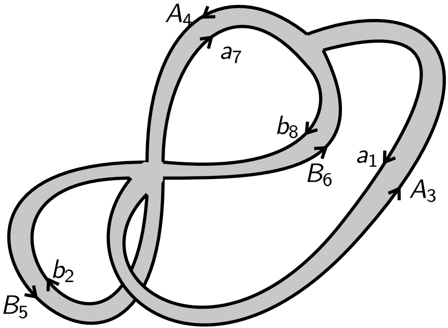


These are all possible strips; the letters are $a_1 b_2 A_3 A_4 B_5 B_6 + a_7 b_8$.

Surfaces in a wedge of loops

Pick strips that contain every letter once, and glue up:

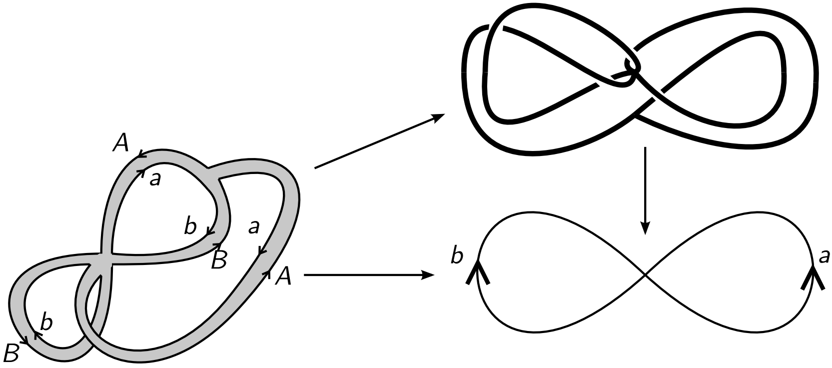




Boundary is $a_1 b_2 A_3 A_4 B_5 B_6 + a_7 b_8$. Note $\chi(S) = 2 - 4 = -2$.

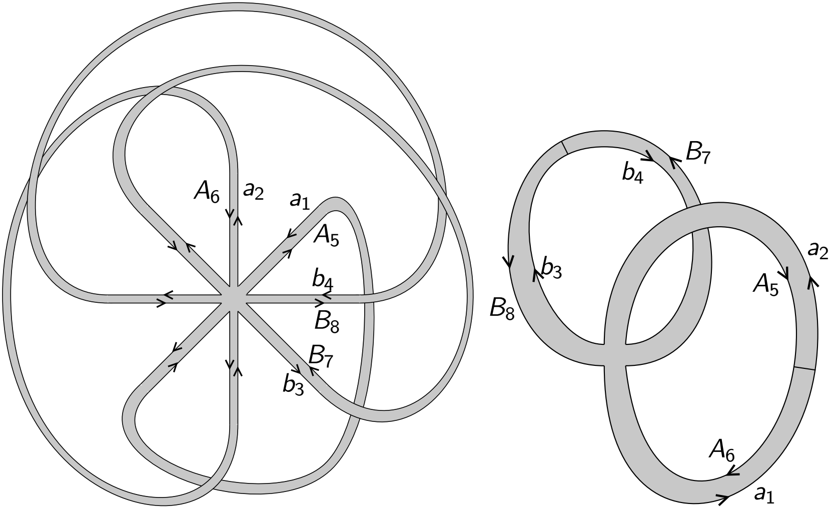
How to map skinny surfaces

We just built a skinny surface with labels. The labels instruct us how to get a map into the wedge of loops:



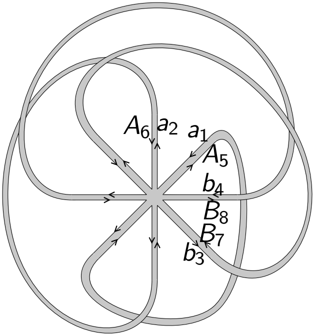
Comparing skinny surfaces

There are multiple ways to pair up the letters to get skinny surfaces with a set boundary. Both of these pairings give surfaces with boundary $aabbAABB$.

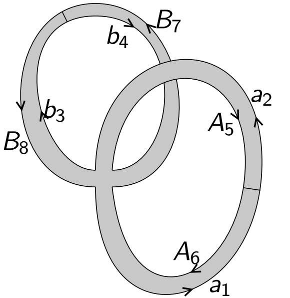


Comparing surfaces in simple spaces

But some surfaces are better than others.



$\chi(S) = 1 - 4 = -3$



$\chi(S) = 3 - 4 = -1$

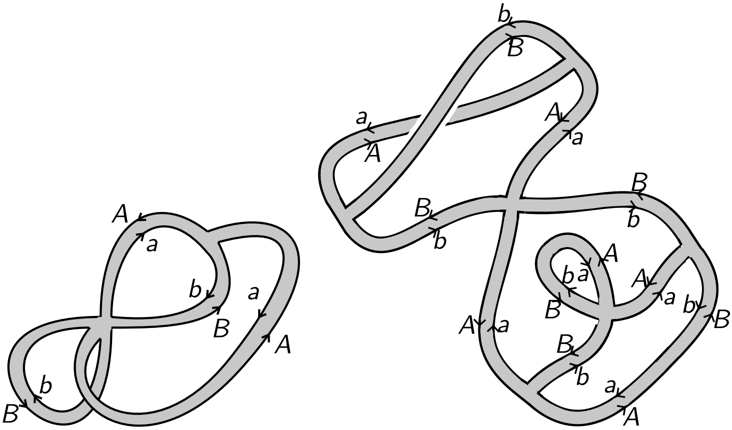
Minimal surfaces

Main question: given some loops, find the surface with minimal Euler characteristic with them as boundary.

Better question: allow the surfaces to wrap around the loops n times, and find the surface with minimal $-\chi(S)/2n$.

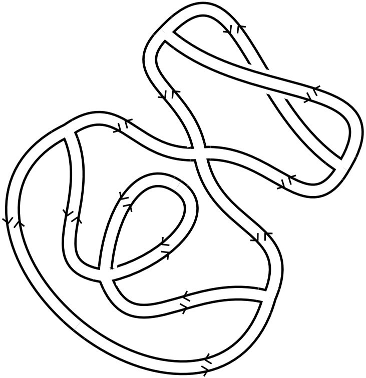
Minimal surfaces

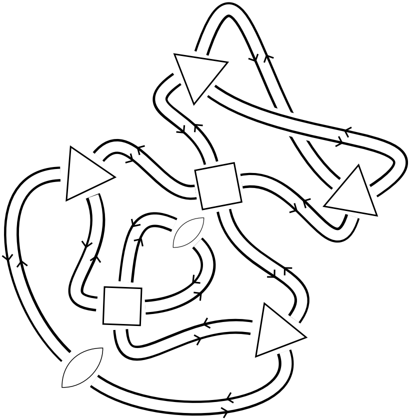
Example: The best surface which wraps around $abAABB + ab$ once has $-\chi(S)/2 = 1$. The most *efficient* surface wraps $n = 3$ times around, and has $-\chi(S)/(2n) = 2/3$.



Finding minimal surfaces

Any skinny surface for $abAABB + ab$ can be broken into pieces:
There are only finitely many kinds of pieces.





Linear programming

- Consider the vector space V over $\mathbb{Q}$ spanned by the pieces (rectangles and polygons).
- The conditions that they can glue up are linear, so there is a subspace of vectors representing skinny surfaces, and a polyhedron of vectors representing skinny surfaces mapping with degree 1.
- Euler characteristic is a linear function
- So we can find a most efficient surface using linear programming.

Linear programming example

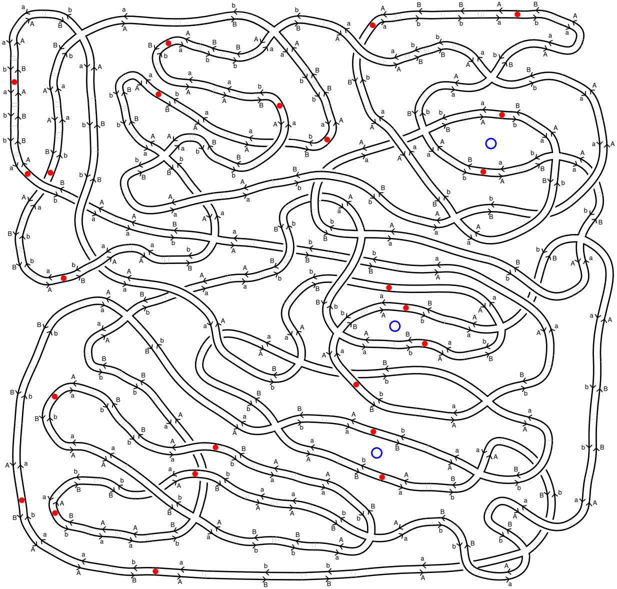
We can compute with long loops; the most efficient surface with boundary:

BBABAbaaBBAbAABBABBaBaaabAbbAbbABabAbbbABAAbbaaBaaBabbABabaB

has $-\chi(S)/2n = 16739/7863 \approx 2.129$.

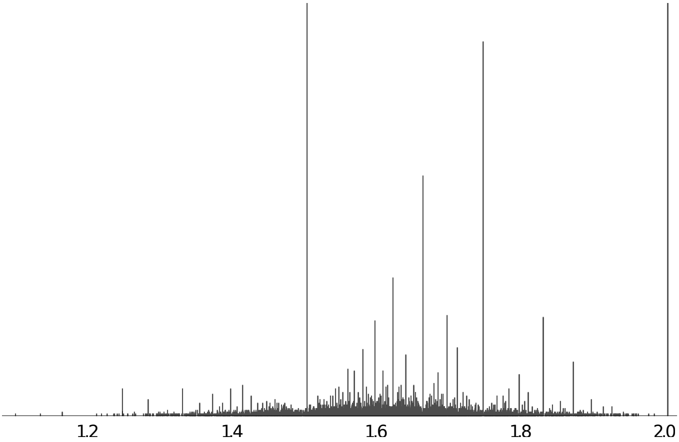
Linear programming example

Here is a big surface that is most-efficient for its boundary:



Random loops

Let's pick lots of random loops, and record the value $\min_S -\chi(S)/2n$ for each. Then we plot the values in a histogram.



Formal definition

Given a group G and $g \in [G, G]$, we say that $f : S \to K(G,1)$ is *admissible* for g if the diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow{f} & K(G,1) \\ \uparrow & & \uparrow \gamma \\ \partial S & \xrightarrow[\partial f]{} & S^1 \end{array}$$

and $\partial f_*[\partial S] = n(S,f)[S^1]$ where $\gamma : S^1 \to K(G,1)$ represents g.
We define:

$$\mathrm{scl}(g) = \inf_{(S,f)} \frac{-\chi(S)}{2n(S,f)}$$

If the infimum is realized by a surface map, the surface is *extremal*.

Group theoretic connection

Given $g \in [G, G]$, the *commutator length* $\mathrm{cl}(g)$ is the least number of commutators in a product that equals g.

We define the *stable commutator length*

$$\mathrm{scl}(g) = \lim_{n \to \infty} \frac{\mathrm{cl}(g^n)}{n}$$

Example:

- $\mathrm{cl}([a, b]) = 1$, so $\mathrm{scl}([a, b]) \leq 1$.
- In fact, $\mathrm{scl}([a, b]) = 1/2$.

Group theoretic connection

Proposition (Calegari)

The definitions agree; i.e.

$$\lim_{n\to\infty} \frac{\mathrm{cl}(g^n)}{n} = \mathrm{scl}(g) = \inf_{(S,f)} \frac{-\chi^-(S)}{2n(S,f)}$$

So for example, a high power of

BBABAbaaBBAbAABBABBaBaaabAbbAbbABabAbbbABAAbbaaBaaBabbABabaB

can be written using, on average, 2.129 commutators per word.

Rationality

Theorem (Calegari)

For F a free group, $g \in [F, F]$, $\mathrm{scl}(g)$ is rational, and there is an extremal surface for g.

Proof.

This talk! ∎

Other facts

Theorem (Calegari)

In a free group, the image of scl *contains every denominator.*

Theorem (Calegari-W)

Let v be a random word of length n in a free group of rank k. As $n \to \infty$, $\text{scl}(v) \to \frac{\log(2k-1)n}{6\log(n)}$ *in probability.*

Theorem (Zhuang)

There is a finitely presented group (one of Thompson's groups) containing elements with transcendental scl*.*

Conjecture

scl *is rational in hyperbolic groups.*

Even more facts

Theorem (W)

There is an efficient algorithm to compute scl *in free products of cyclic groups.*

Theorem (Lvzhou (Joe) Chen)

scl *is piecewise rational linear in groups acting on a trees with cyclic edge and vertex stabilizers, and there are extremal surfaces in certain cases in Baumslag-Solitar groups.*

Chen combines and generalizes essentially all previous known results about computing scl into one theorem.

References

- Calegari, Danny. *scl*. MSJ Memoirs, 20. Mathematical Society of Japan, Tokyo, 2009.
- Chen, Lvzhou. *Scl in graphs of groups*. arXiv:1904.08360
- `aldenwalker.org`